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LXXVI. *The Diffraction of Electromagnetic Waves from an Electrical Point Source round a Finitely Conducting Sphere, with Applications to Radiotelegraphy and the Theory of the Rainbow* *.—Part II.

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9. *The Secondary Field considered as the Superposition of an Infinity of Waves originating after* 1, 2, 3 . . . . *Reflections.*

IN Part I. of this paper the diffraction round a sphere of waves originating from a point source was considered starting with the harmonic series (1). Thereupon a transformation of this series was affected with the aid of a contour integral, resulting in a series which converges rather rapidly in the shadow region, while the convergence in the "lit" part of space is not appropriate for numerical evaluation. As in many physical applications a knowledge of the wave-field in the "lit" part of space is of importance, we now proceed to an entirely different attack on the problem; while again starting

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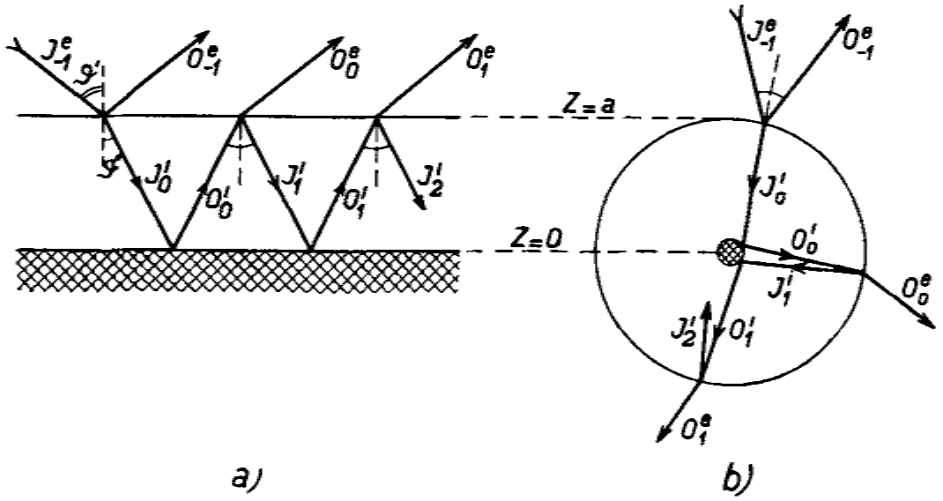
with the rigorous series (1), the fundamental idea used here is to split up each of the terms of (1) into a geometrical series, after which the corresponding terms of these geometrical series are combined. This splitting-up process is based on the physical conception of radial waves being reflected several times between the surface and the centre of the sphere. Thereupon each of these new series of radial waves is written in the form of a multiple exponential integral. It next appears that the position of the saddle points of these integrals determines the geometrical points at the surface of the sphere, where a ray joining the point source and

Fig. 12.



(*a*) infinite slab, (*b*) sphere, showing the different elementary waves.

the observer is refracted and reflected. Thus our saddle point treatment of wave functions appears to be identical with the application of Fermat's principle. The second order approximation of these integrals with the method of stationary phase yields the amplitude of these rays whose geometrical form was given by the above-mentioned saddle points. These rays correspond exactly with the classical geometrical ray treatment of the rainbow which originates from Descartes.

In order to illustrate the fundamental idea of the splitting up of the wave function in successively refracted and reflected waves we treat in this paragraph an analogous simple problem.

Consider the problem with two plane boundaries represented by fig. 12 *a*, where an electrical vertical dipole is situated at $z=b$, $\rho=0$ in the medium 1 with propagation constant k_1. Medium 2, with propagation constant k_2, consists of an infinite slab with *plane* boundaries at $z=a$ and $z=0$. Below $z=0$ we assume a third medium of infinite conductivity ($k_3=\infty$), so that all waves penetrating into the second medium are totally reflected at the boundary surface $z=0$. Later on we shall see that in the spherical analogy (fig. 12 *b*) we again find a total reflexion, but there against the centre of the sphere. This plane problem is easily solved with the well-known integral method of Sommerfeld, and leads, for a primary field $e^{ik_1 R}/(ik_1 R)$, to a secondary field in the first medium, given by

$$\Pi_{\text{sec}} = \frac{1}{ik_1} \cdot \int_0^\infty \frac{\lambda\, d\lambda}{\sqrt{\lambda^2-k_1^2}} \cdot R_\rho(\lambda) \cdot J_0(\lambda\rho) \cdot e^{-\sqrt{\lambda^2-k_1^2}(z+b-2a)}, \quad \ldots \quad (60)$$

where

$$R_\rho = \frac{\frac{\sqrt{\lambda^2-k_1^2}}{k_1^2} - \frac{\sqrt{\lambda^2-k_2^2}}{k_2^2} \cdot Tgh(\sqrt{\lambda^2-k_2^2}a)}{\frac{\sqrt{\lambda^2-k_1^2}}{k_1^2} + \frac{\sqrt{\lambda^2-k_2^2}}{k_2^2} \cdot Tgh(\sqrt{\lambda^2-k_2^2}a)}, \quad (61)$$

and ρ is the horizontal distance from point source to receiver.

If, with Sommerfeld and Weyl, (60) is considered as the superposition of plane waves, λ has the meaning of $k_1 \sin \vartheta'$, where ϑ' is the angle a particular plane wave makes with the vertical. The fundamental idea of this paragraph is to write (61) as the geometrical series

$$R_\rho = R(\lambda) + \sum_{K=0}^{\infty} \{1+R(\lambda)\} \cdot R(\lambda)^K \{1-R(\lambda)\} \cdot e^{-2(K+1)a\sqrt{\lambda^2-k_2^2}}, \quad (62)$$

so that, when inserted into (60), the secondary field in the first medium becomes the sum

$$\Pi_{\text{sec}} = O^e_{-1} + O_0{}^e + O_1{}^e + O_2{}^e + O_3{}^e + \ldots, \quad \ldots \quad (63)$$

where all the O^e_K mean outgoing waves in the first medium, represented by *

$$\left\{\begin{aligned} O^e_{-1} &= \frac{1}{ik_1}\cdot\int_0^\infty \frac{\lambda\,d\lambda}{\sqrt{\lambda^2-k_1^2}}\cdot R(\lambda)\cdot J_0(\lambda\rho)\cdot e^{-\sqrt{\lambda^2-k_1^2}(z+b-2a)}. \qquad \ldots \quad (64a)\\ O^e_K &= \frac{1}{ik_1}\cdot\int_0^\infty \frac{\lambda\,d\lambda}{\sqrt{\lambda^2-k_1^2}}\cdot\{1+R(\lambda)\}\cdot R(\lambda)^K\cdot\{1-R(\lambda)\}\\ &\quad\cdot J_0(\lambda\rho)\cdot e^{-\sqrt{\lambda^2-k_1^2}(z+b-2a)-2(K+1)a\sqrt{\lambda^2-k_2^2}}, \quad (K\geqq 0), \quad (64b)\end{aligned}\right.$$

and where R (λ) has the usual form

$$R(\lambda)=\frac{k_2^2\sqrt{\lambda^2-k_1^2}-k_1^2\sqrt{\lambda^2-k_2^2}}{k_2^2\sqrt{\lambda^2-k_1^2}+k_1^2\sqrt{\lambda^2-k_2^2}}, \quad \ldots \quad (65)$$

meaning physically the Fresnel reflexion coefficient for the waves incident at an angle ϑ', *cf.* (21). The physical meaning of (64 *b*) becomes at once clear if in the integral we put

$$\lambda=k_1\sin\vartheta'=k_2\sin\vartheta'',$$

so that we obtain

$$O^e_K=\int_0^{\pi/2-i\infty}\sin\vartheta'\,d\vartheta'\cdot\{1+R(\lambda)\}\cdot R(\lambda)^K\cdot\{1-R(\lambda)\}\cdot J_0(\lambda\rho)\cdot e^{ik_1\cos\vartheta'(z+b-2a)+2ik_2(K+1)a\cos\vartheta''}. \quad (66)$$

The form of (66) is easily interpreted as an outgoing wave in the first medium (see fig. 12 *a*), originating from the primary incident wave I^e_{-1} in the first medium, which first is refracted as the ingoing wave I^i_0 in the second medium, this being afterwards totally reflected at $z=0$, thus becoming O^i_0; this outgoing wave, when reaching $z=a$, is split up in the refracted wave O^e_0 in the first medium and the reflected ingoing wave I^i_1 in the second medium. This process is repeated till finally we obtain O^e_K as the outgoing wave in the first medium, resulting from waves in the second medium which have been reflected up and down in the second medium with K reflexions at $z=a$ and K+1 at $z=0$. It will be clear that the total secondary field Π_{sec}, in accordance with (63), is obtained by superposing all the outgoing waves characterized by K$=-1$, 0, $+1$, $+2$ .... etc. in the

* For the symmetry of the notation we give the incident and first reflected wave the suffix -1.

first medium. The notations are chosen such that O and I relate resp. to the outgoing and ingoing waves, whereas the index e or i denotes the first or second medium.

The above physical consideration now makes the form of (66) quite obvious, because the integrand contains the factors

$$\{1+\mathrm{R}(\lambda)\}.\mathrm{R}(\lambda)^{\mathrm{K}}.\{1-\mathrm{R}(\lambda)\}$$

and

$$e^{2ik_2(\mathrm{K}+1)a\cos\vartheta''},$$

the first of which represents the attenuation $1+\mathrm{R}(\lambda)$ at the first refraction, the attenuations $\mathrm{R}(\lambda)^{\mathrm{K}}$ due to the K reflexions at $z=a$, and the attenuation $1-\mathrm{R}(\lambda)$ at the final refraction into the first medium.

The second factor represents the phase retardation and, for complex k_2, the attenuation due to the fact that the wave has travelled $\mathrm{K}+1$ times up and down in the second medium, the form of this factor being quite similar to that obtained in the derivation of the phase retardation in Bragg's well-known formula of crystal reflexion.

10. *The "Spherical Reflexion Coefficients."*

In the last paragraph when we split up the secondary field into an infinity of components we were naturally lead to introduce the ordinary Fresnel reflexion coefficient which belongs to the reflexion of a plane wave against a plane boundary. Before considering in more detail the analogous splitting up of the waves in our spherical problem we shall have to investigate first the extension of the simple Fresnel reflexion coefficient for the reflexion of an incident spherical wave

$$\zeta_n^{(2}(k_1r).\mathrm{P}_n(\cos\vartheta)$$

against the concentric sphere $r=a$ separating the first medium (k_1) from the second medium (k_2).

We can write the reflected wave in the first medium as

$$\mathrm{R}_{11}.\zeta_n^{(2)}(k_1a).\frac{\zeta_n^{(1)}(k_1r)}{\zeta_n^{(1)}(k_1a)}.\mathrm{P}\ (\cos\vartheta),\ .\quad .\quad (67a)$$

and the refracted wave in the second medium as

$$\mathrm{R}_{12}.\zeta_n^{(2)}(k_1a).\frac{\zeta_n^{(2)}(k_2r)}{\zeta_n^{(2)}(k_2a)}.\mathrm{P}_n(\cos\vartheta).\quad .\quad .\quad (67b)$$

These forms have been chosen such that, *e. g.*, R_{11} means the ratio of the amplitudes of the reflected and incident waves at $r=a$, and similarly for R_{12}.

The boundary conditions, $\frac{\partial}{\partial r}(r\Pi)$ and $k^2\Pi$ continuous at $r=a$, now fully determine the "*spherical reflexion and refraction coefficients*" R_{11} and R_{12}, which in our case, where we assume the electric field to be in the plane of incidence (electrical dipole), become

$$R_{11} = \frac{-\left[\frac{1}{x}\frac{d}{dx}\log\{x\zeta_n^{(2)}(x)\}\right]_{x=k_1a} + \left[\frac{1}{x}\frac{d}{dx}\log\{x\zeta_n^{(2)}(x)\}\right]_{x=k_2a}}{\left[\frac{1}{x}\frac{d}{dx}\log\{x\zeta_n^{(1)}(x)\}\right]_{x=k_1a} - \left[\frac{1}{x}\frac{d}{dx}\log\{x\zeta_n^{(2)}(x)\}\right]_{x=k_2a}}, \quad . \; . \; . \quad (68a)$$

$$R_{12} = \frac{k_1^2}{k_2^2}(1+R_{11}). \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (68b)$$

It is of interest to contrast the more physical factor R_{11}, which contains ζ functions only, with the formally used R_n of (2), which contains both ζ and ψ functions.

11. *The Splitting-up of the Secondary Field into an Infinity of Spherical Waves originating after* 1, 2, 3 . . . *Reflexions at the Centre and the Surface of the Sphere.*

After having derived the ideas of the spherical reflexion and refraction coefficients we are now able to interpret the spherical case as an analogue of the plane problem of § 9. To this end we follow again the waves on their courses. According to (3 *b*) the primary field for $r<b$ contains the standing waves $\psi_n(k_1r)P_n(\cos\vartheta)$. As

$$\psi_n(k_1r) = \tfrac{1}{2}\zeta_n^{(1)}(k_1r)+\tfrac{1}{2}\zeta_n^{(2)}(k_1r),$$

we can consider the primary field as containing both outgoing $\zeta^{(1)}$ and ingoing $\zeta^{(2)}$ waves. The outgoing part will be presently left aside. The ingoing part is given by

$$I^e_{-1} = \tfrac{1}{2}\sum_{n=0}^{\infty}(2n+1).\zeta_n^{(1)}(k_1b).\zeta_n^{(2)}(k_1r).P_n(\cos\vartheta).$$

As explained in § 10 each of the components, say $\zeta_n^{(2)}(k_1r)P_n(\cos\vartheta)$, is first at $r=a$ split up into a reflected

wave in the first medium and a refracted wave in the second medium, with the reflexion and refraction coefficients R_{11} and R_{12} resp., thus, after summation over n, giving rise to the waves O^e_{-1} in the first medium and I^i_0 in the second (see fig. 12 *b*). Each of the components of the latter, say

$$\tfrac{1}{2}(2n+1).R_{12}.\zeta_n^{(1)}(k_1b)\zeta_n^{(2)}(k_1a).\frac{\zeta_n^{(2)}(k_2r)}{\zeta_n^{(2)}(k_2a)}.P_n(\cos\vartheta),$$

would, however, cause a singularity at $r=0$, due to the factor $\zeta_n^{(2)}(k_2r)$, which can only be compensated by an outgoing wave

$$\tfrac{1}{2}(2n+1).R_{12}.\zeta_n^{(1)}(k_1b)\zeta_n^{(2)}(k_1a).\frac{\zeta_n^{(1)}(k_2r)}{\zeta_n^{(2)}(k_2a)}.P_n(\cos\vartheta),$$

having the same amplitude. This fact can be interpreted as a total reflexion of the ingoing wave I^i_0 against an infinitely small sphere at the centre. This is, moreover, confirmed by the fact that the limit for $a=0$ of the spherical reflexion coefficient (68 *a*) becomes unity independently of the values of k_1 and k_2. Hence it can be said that the incident wave in the sphere is totally reflected against its centre analogous to the reflexion against the plane $z=0$ of the plane problem of § 9.

The summation over n of these totally reflected waves results in the outgoing wave O^i_0, but still being in the second medium. Following this wave further, it will arrive at the inside of the boundary $r=a$, where it will be divided into an ingoing reflected wave I^i_1 in the second medium (see fig. 12 *b*) and an outgoing refracted wave O^e_0 in the first medium, where it has to be noted that now this reflexion and refraction occur with the factors R_{22} and R_{21}, derivable from R_{11} and R_{12} by interchanging k_1 and k_2. This process repeats itself indefinitely in a fully analogous way as described in the plane problem of § 9 (compare fig. 12 *a* and fig. 12 *b*); therefore we have chosen the notations for the different waves quite the same as in § 9.

We thus obtain instead of (63) and (64) the following final formulæ:

for the total field in the first medium

$$\Pi_{\text{tot}} = I^e_{-1}+O^e_{-1}+O_0{}^e+O_1{}^e+O_2{}^e+O_3{}^e+\ldots\ (r>a), \quad (69\,a)$$

and similarly for the total field in the second medium

$$\Pi_{\text{tot}} = \mathrm{I}_0{}^i + \mathrm{O}_0{}^i + \mathrm{I}_1{}^i + \mathrm{O}_1{}^i + \mathrm{I}_2{}^i + \mathrm{O}_2{}^i + \mathrm{I}_3{}^i + \mathrm{O}_3{}^i + \ldots (r<a), \quad \ldots \quad (69\,b)$$

where

$$\mathrm{I}^e_{-1} = \begin{cases} \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\zeta_n^{(2)}(k_1 b)\zeta_n^{(1)}(k_1 r)\mathrm{P}_n(\cos\vartheta), & (r>b), \\ \qquad \ldots \quad (70\,a) \\ \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\zeta_n^{(1)}(k_1 b)\zeta_n^{(2)}(k_1 r)\mathrm{P}_n(\cos\vartheta), & (r<b), \\ \qquad \ldots \quad (70\,b) \end{cases}$$

$$\mathrm{O}^e_{-1} = \tfrac{1}{2}\sum_{u=0}^{\infty}(2n+1)\mathrm{R}_{11}\zeta_n^{(1}(k_1 b)\zeta_n^{(2)}(k_1 a)\cdot\frac{\zeta_n^{(1)}(k_1 r)}{\zeta_n^{(1)}(k_1 a)}\cdot\mathrm{P}_n(\cos\vartheta), \quad \ldots \quad (70\,c)$$

$$\mathrm{O}^e_{\mathrm{K}} = \tfrac{1}{2}\sum_{n=0}^{\infty}(2n+1)\mathrm{R}_{12}\mathrm{R}^{\mathrm{K}}_{22}\mathrm{R}_{21}\cdot\zeta_n^{(1)}(k_1 b)\zeta_n^{(2)}(k_1 a) \cdot\left\{\frac{\zeta_n^{(1)}(k_2 a)}{\zeta_n^{(2)}(k_2 a)}\right\}^{\mathrm{K}+1}\cdot\frac{\zeta_n^{(1)}(k_1 r)}{\zeta_n^{(1)}(k_1 a)}\cdot\mathrm{P}_n(\cos\vartheta), \quad (\mathrm{K}\geqq 0), \quad (70\,d)$$

$$\mathrm{I}^i_{\mathrm{K}} = \tfrac{1}{2}\sum_{n=0}^{\infty}(2n+1)\mathrm{R}_{12}\mathrm{R}^{\mathrm{K}}_{22}\cdot\zeta_n^{(1)}(k_1 b)\zeta_n^{(2)}(k_1 a) \cdot\left\{\frac{\zeta_n^{(1)}(k_2 a)}{\zeta_n^{(2)}(k_2 a)}\right\}^{\mathrm{K}}\cdot\frac{\zeta_n^{(2)}(k_2 r)}{\zeta_n^{(2)}(k_2 a)}\cdot\mathrm{P}_n(\cos\vartheta), \quad (\mathrm{K}\geqq 0), \quad (70\,e)$$

$$\mathrm{O}^i_{\mathrm{K}} = \tfrac{1}{2}\sum_{n=0}^{\infty}(2n+1)\mathrm{R}_{12}\mathrm{R}^{\mathrm{K}}_{22}\cdot\zeta_n^{(1)}(k_1 b)\zeta_n^{(2)}(k_1 a) \cdot\left\{\frac{\zeta_n^{(1)}(k_2 a)}{\zeta_n^{(2)}(k_2 a)}\right\}^{\mathrm{K}}\cdot\frac{\zeta_n^{(1)}(k_2 r)}{\zeta_n^{(2)}(k_2 a)}\cdot\mathrm{P}_n(\cos\vartheta), \quad (\mathrm{K}\geqq 0). \quad (70f)$$

In deriving these formulæ we started, according to (70 *b*), with the ingoing part only of the primary field. It can, however, easily be shown that, in order to satisfy the boundary conditions, the other, outgoing, part of the primary field must be accompanied by a secondary field which just cancels it, so that our solution (70) gives the total field, satisfying the boundary conditions and having as only singularity the point source at Q.

It may be finally remarked that the physical arguments used in this section are equivalent to the development of the denominator of the general R_n of (2) into a geometrical

series, just as in the plane case of § 9. Thus we get instead of (62) for R_n:

$$R_n = -\tfrac{1}{2}\frac{\zeta_n^{(1)}(k_1a)}{\psi_n(k_1a)} + \tfrac{1}{2}\frac{\zeta_n^{(2)}(k_1a)}{\psi_n(k_1a)} \cdot R_{11} + \tfrac{1}{2}\frac{\zeta_n^{(2)}(k_1a)}{\psi_n(k_1a)} \cdot \sum_{K=0}^{\infty} (1+R_{11})R_{22}^K(1+R_{22}) \cdot \left\{\frac{\zeta_n^{(1)}(k_2a)}{\zeta_n^{(2)}(k_2a)}\right\}^{K+1}. \quad (71)$$

As in § 9, we here again encounter the two factors

$$(1+R_{11})R_{22}^K(1+R_{22}) = R_{12}R_{22}^K R_{21},$$

representing the attenuations caused by the first refraction, the K reflexions, and the last refraction, while the factor

$$\left\{\frac{\zeta_n^{(1)}(k_2a)}{\zeta_n^{(2)}(k_2a)}\right\}^{K+1}$$

represents the phase retardation and attenuation as a result of the waves having travelled K+1 times to and fro between the centre and the surface of the sphere.

It is of importance to note that the factors

$$\zeta_n^{(1)}(k_2a)/\zeta_n^{(2)}(k_2a)$$

tend towards zero for both $k_2 \to \infty$ and $a \to \infty$, proving the fact that (*a*) for a totally reflecting sphere of finite size and (*b*) for a finitely conducting sphere but of infinite radius, and therefore tending to a plane, no reflected waves from the inside will be present. Limiting ourselves to the latter case ($a \to \infty$) we note that, even with the slightest absorption, [$Jm(k_2a) \to 0$], only I^e_{-1} and O^e_{-1} are left as components outside the sphere. The part O^e_{-1}, treated after the method of § 5, leads to the same result as obtained there, viz., the classical expression for the plane case in the form of the λ integral for Π_{sec}.

12. *Reduction of the Subsidiary Waves* O^e_K *to a Sum of Residues.*

In § 3 formula (7) it was shown, following Watson, that Π_{tot} could be transformed into the sum of a closed contour integral and an open line integral along the imaginary axis. This inhomogeneous character of the result was due to the fact that the integrand was not an even function of $n-\frac{1}{2}$. However, the splitting up of Π_{tot} into an infinity of subsidiary waves O^e_K (as affected

in § 11) enables us to write each of these waves as a closed contour integral only, encircling poles of the order K+2, so that the total field thus becomes rigorously a sum of residues, without any additional open line integral.

This transformation is based on the following consideration. As each of the subsidiary waves O^e_K (as given by (70 *d*)) is represented by a sum over n, n being integral, and as we want to transform the sum into an integral with integrand, which, apart from the factor $2n+1$, is even in $n-\frac{1}{2}$, we are at liberty to continue the functions of n for non-integral values of this variable in different ways before we transform the sum into an integral. Incidentally the result thus obtained expresses each of the subsidiary waves O^e_K itself again as an infinite sum of elementary waves $O^e_{K,m}$, where m denotes the number of times each of these waves has travelled round the sphere (compare § 3, equation (9)).

In order to effect this plan we first notice the identity

$$\{\zeta_n^{(1)}(k_2a)\}^{K+1}.P_n(\cos\vartheta) = \{\zeta_n^{(2)}(k_2ae^{-i\pi})\}^{K+1} .P_n\{\cos(\vartheta-K\pi-\pi)\},$$

with which, *e. g.*, (70 *d*) can be written

$$O^e_K = \sum_{n=0}^{\infty}(2n+1).g(n).P_n\{\cos(\vartheta-K\pi-\pi)\}, \quad (72)$$

with

$$g(n) = \tfrac{1}{2}R_{12}R_{22}^K R_{21}.\zeta_n^{(1)}(k_1b)\zeta_n^{(2)}(k_1a) \left\{\frac{\zeta_n^{(2)}(k_2ae^{-i\pi})}{\zeta_n^{(2)}(k_2a)}\right\}^{K+1}.\frac{\zeta_n^{(1)}(k_1r)}{\zeta_n^{(1)}(k_1a)}.$$

As a consequence of the relation

$$\zeta^{(2)}_{-n-1/2}(z) = e^{\pm in\pi}.\zeta^{(1)}_{n-1/2}(z)$$

it follows that $g(n-\frac{1}{2})$ is built up of factors each even in n. This is also true for the "spherical reflexion coefficients" such as R_{12}, whereas the formerly used coefficient R (equation (2)) is *not* an even function of n.

The fact that $g(n-\frac{1}{2})$ is even in n enables us, following the method of § 3, to transform the sum (72) first to the integral

$$O^e_K = \frac{1}{i}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{n\,dn}{\cos(n\pi)}.g(n-\tfrac{1}{2}).P_{n-1/2}\{\cos(\vartheta-K\pi-2\pi)\}. \quad . \quad . \quad . \quad (73)$$

As, further, along the whole path of integration $Jm(n)>0$, we may develop the factor $\{\cos(n\pi)\}^{-1}$ as

$$\frac{1}{\cos(n\pi)} = 2\sum_{m=0}^{\infty}(-1)^m . e^{in\pi(2m+1)},$$

so that (73) becomes

$$O^e_K = \sum_{m=0}^{\infty} O^e_{K,m}, \quad . \quad . \quad . \quad . \quad . \quad (74)$$

where

$$O^e_{K,m} = \frac{(-1)^m}{i}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} n\,dn\, e^{in\pi(2m+1)} . R_{12}R^K_{22}R_{21} \cdot \zeta^{(1)}_{n-1/2}(k_1b)\zeta^{(2)}_{n-1/2}(k_1a) \cdot \left\{\frac{\zeta^{(2)}_{n-1/2}(k_2ae^{-i\pi})}{\zeta^{(2)}_{n-1/2}(k_2a)}\right\}^{K+1} \cdot \frac{\zeta^{(1)}_{n-1/2}(k_1r)}{\zeta^{(1)}_{n-1/2}(k_1a)} . P_{n-1/2}\{\cos(\vartheta - K\pi)\}. \quad . \quad . \quad (75)$$

This integral (75) can be closed by a half-circle of infinite radius above the real axis, so that its value can be represented by the sum of residues of poles n_s of the order $K+2$. These poles are the zeros of $M_{n-\frac{1}{2}}=0$, where

$$M_n = \left[\frac{1}{x}\cdot\frac{d}{dx}\log\{x\zeta^{(1)}_n(x)\}\right]_{x=k_1a} - \left[\frac{1}{y}\cdot\frac{d}{dy}\log\{y\zeta^{(2)}_n(y)\}\right]_{y=k_2a}, \quad (76)$$

which occurs in (68 *b*) and is only slightly different from the expression N_n defined by (6). Next making use of the asymptotic formula (12), each of the residues is seen to have as exponential part the factor

$$e^{in_s(\vartheta+2m\pi)},$$

showing, finally, that each of the terms $O^e_{K,m}$ represents an elementary wave which has been *reflected* K *times*, *refracted* 2 *times, and diffracted m times* (having encircled the sphere *m* times) before arriving at the receiver.

In considering the theory of the rainbow the most important elementary waves are $O^0_{1,0}$ and $O^0_{2,0}$, whereas in the *radio case* the elementary waves $I^e_{1,0}$ and $O^e_{-1,0}$ form the essential part. A further analysis shows that the

sum $I^e_{-1,0}+O^e_{-1,0}$ can also be transformed to a sum of residues of poles of order unity, yielding

$$I^e_{-1,0}+O^e_{-1,0}=\frac{4\pi i}{(k_1a)^3}\cdot\sum_{s=0}^{\infty}\frac{n_s\cdot e^{in_s\pi}}{\left(\frac{\partial M_{n-1/2}}{\partial n}\right)_{n=n_s}}$$

$$\cdot\frac{\zeta^{(1)}_{n_s-1/2}(k_1b)\zeta^{(1)}_{n_s-1/2}(k_1r)}{\{\zeta^{(1)}_{n_s-1/2}(k_1a)\}^2}\cdot P_{n_s-1/2}\{\cos(\pi-\vartheta)\},\quad(77,$$

and it is on this expression that we will later on base our numerical calculations in the radio case (see § 17).

13. *Transformation of the Elementary Wave* $O^e_{K,m}$ *into a* $(2K+8)$*-dimensional Exponential Integral, the Saddle Point of which determining a* K*-times internally reflected Goemetrical Optical Ray.*

In this paragraph we will develop a totally different aspect of our problem, leaving for the present the residue method. Here we will transform the general elementary wave $O^e_{K,m}$ into a more dimensional exponential integral. It will be seen that the position of its saddle point determines at once the only geometrical optical ray connecting the transmitter and receiver, which has been reflected internally K times. Moreover, a further development of the exponent round the saddle point provides us with an expression for the amplitudes of this ray. Performing this transformation for all values of K we thus succeed in writing the complete solution of our problem as the sum of an infinite number of geometrical optical rays.

In order to affect this transformation of (75) we first note that the denominator only contains $\zeta^{(1)}$ and $\zeta^{(2)}$ functions, each of which possesses an exponential character which we can annul by multiplying the nominator and denominator each by

$$\zeta^{(2)}_{n-1/2}(k_1a).\{\zeta^{(1)}_{n-1/2}(k_2a)\}^{K+1},$$

resulting in a denominator which only contains factors of the form $\zeta^{(1)}_{n-1/2}(x)\,.\,\zeta^{(2)}_{n-1/2}(x)$, where the exponential character has disappeared. As the same was already true as regards the factors R_{12} R^K_{22} R_{21}, we are now prepared to transform each of the factors remaining

in the nominator into an exponential integral by the aid of the fundamental expressions

$$\zeta^{(1)}_{n-1/2}(z) = \frac{1}{\sqrt{2\pi z}} \cdot \int_{-\eta+i\infty}^{\eta-i\infty} e^{iz\cos\tau + in(\tau-\pi/2)}\, d\tau, \quad (78\,a)$$

$$\zeta^{(2)}_{n-1/2}(z) = -\frac{1}{\sqrt{2\pi z}} \cdot \int_{\pi-\eta+i\infty}^{\eta-\pi-i\infty} e^{-iz\cos\tau - in(\tau-\pi/2)}\, d\tau, \quad (78\,b)$$

where $-\arg z < \eta < \pi - \arg z$.
Writing finally

$$P_{n-1/2}\{\cos(\vartheta - K\pi)\} = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} e^{(n-1/2)\log\{\cos(\vartheta-K\pi)+i\sin(\vartheta-K\pi)\cos\phi\}}\, d\phi,$$

we obtain, instead of (75),

$$O^e_{K,m} = \frac{(-1)^m}{8\pi^3 i k_1^2 a\sqrt{br}(2\pi i k_2 a)^{K+1}} \cdot \int_{-\infty}^{\infty} dn\, A(n) \cdot \int_{-\pi}^{\pi} d\phi \int d\tau_1 \ldots d\tau_{2K+6} \cdot e^{f^e_{K,m}}, \quad (79)$$

where the non-exponential factor $A(n)$ is given by

$$A(n) = \frac{n \cdot R_{12} R_{22}^K R_{21}}{\zeta^{(1)}_{n-1/2}(k_1 a)\zeta^{(2)}_{n-1/2}(k_1 a) \cdot \{\zeta^{(1)}_{n-1/2}(k_2 a)\zeta^{(2)}_{n-1/2}(k_2 a)\}^{K+1}},$$

while the exponential factor exp. $(f^e_{K,m})$ has as exponent

$$\begin{aligned} f^e_{K,m} = {} & ik_1 b\cos\tau_1 - ik_1 a\cos\tau_2 + ik_2 a \sum_{V=3}^{2K+4} \cos\tau_V \\ & - ik_1 a\cos\tau_{2K+5} + ik_1 r\cos\tau_{2K+6} \\ & + in\left\{\tau_1 - \tau_2 + \sum_{V=3}^{2K+4}\tau_V - \tau_{2K+5} + \tau_{2K+6} + (2m+1)\pi\right\} \\ & + (n-\tfrac{1}{2})\log\{\cos(\vartheta - K\pi) + i\sin(\vartheta - K\pi)\cos\phi\}. \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots \quad (80) \end{aligned}$$

Further, the paths of integration of τ_1, $\tau_3 \ldots \tau_{2K+4}$, τ_{2K+6} are as in (78 *a*), while those of τ_2 and τ_{2K+5} are as in (78 *b*).

The physical character of (79) becomes at once apparent if we consider the position of the multi-dimensional saddle point of exp. $(f^e_{K,m})$, which is given by

$$\frac{\partial f}{\partial \tau_1} = \frac{\partial f}{\partial \tau_2} = \ldots = \frac{\partial f}{\partial \tau_{2K+6}} = \frac{\partial f}{\partial n} = \frac{\partial f}{\partial \phi} = 0,$$

the solutions of which are

$$\begin{cases} n=k_1 b \sin\tau_1=k_1 a \sin\tau_2=k_2 a \sin\tau_3=\ldots=k_2 a \sin\tau_{2K+} \\ \qquad\qquad =k_1 a \sin\tau_{2K+5}=k_1 r \sin\tau_{2K+6}, \quad (81\,a) \\ (\tau_2-\tau_1)+\sum_{A=3}^{2K+4}(\tfrac{1}{2}\pi-\tau_v)+(\tau_{2K+5}-\tau_{2K+6})=\vartheta+2m\pi, \\ \qquad\qquad . \quad . \quad . \quad (81\,b) \\ \phi=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81\,c) \end{cases}$$

If we now identify the variables τ *with the angles as shown in the schematic fig.* 13, *which we consider to lie in the plane* $\phi=0$, *the expressions* (81) *determine exactly the geometrical-optical broken ray, which first leaves the transmitter at* Q, *is thereupon refracted at* A_1, *and next internally reflected at* $A_2, \ldots . A_{K+1}$, *and which finally leaves the sphere as the refracted ray* $A_{K+2}P$. That this is the case follows immediately from the fact that (81 *a*) represents nothing but Snellius's laws for each reflexion and refraction, and (81 *b*) expresses that the total angular deviation of the ray as measured at the centre O is the sum of the angles $(\tau_2-\tau_1)$, $(\tfrac{1}{2}\pi-\tau_2)$, etc.

Having thus determined by the position of the saddle points the geometrical optical rays, the intensity of these rays follows from an approximate evaluation of the integrals (79), but before we proceed we first insert a short consideration on the physical meaning of the first, second, third, and fourth order approximations of integrals of the type of (79).

14. *The Physical Meaning of Approximations of Different Order of Saddle Point Integrals.*

A complex exponential integral,

$$\int e^{f(z)}\,dz, \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

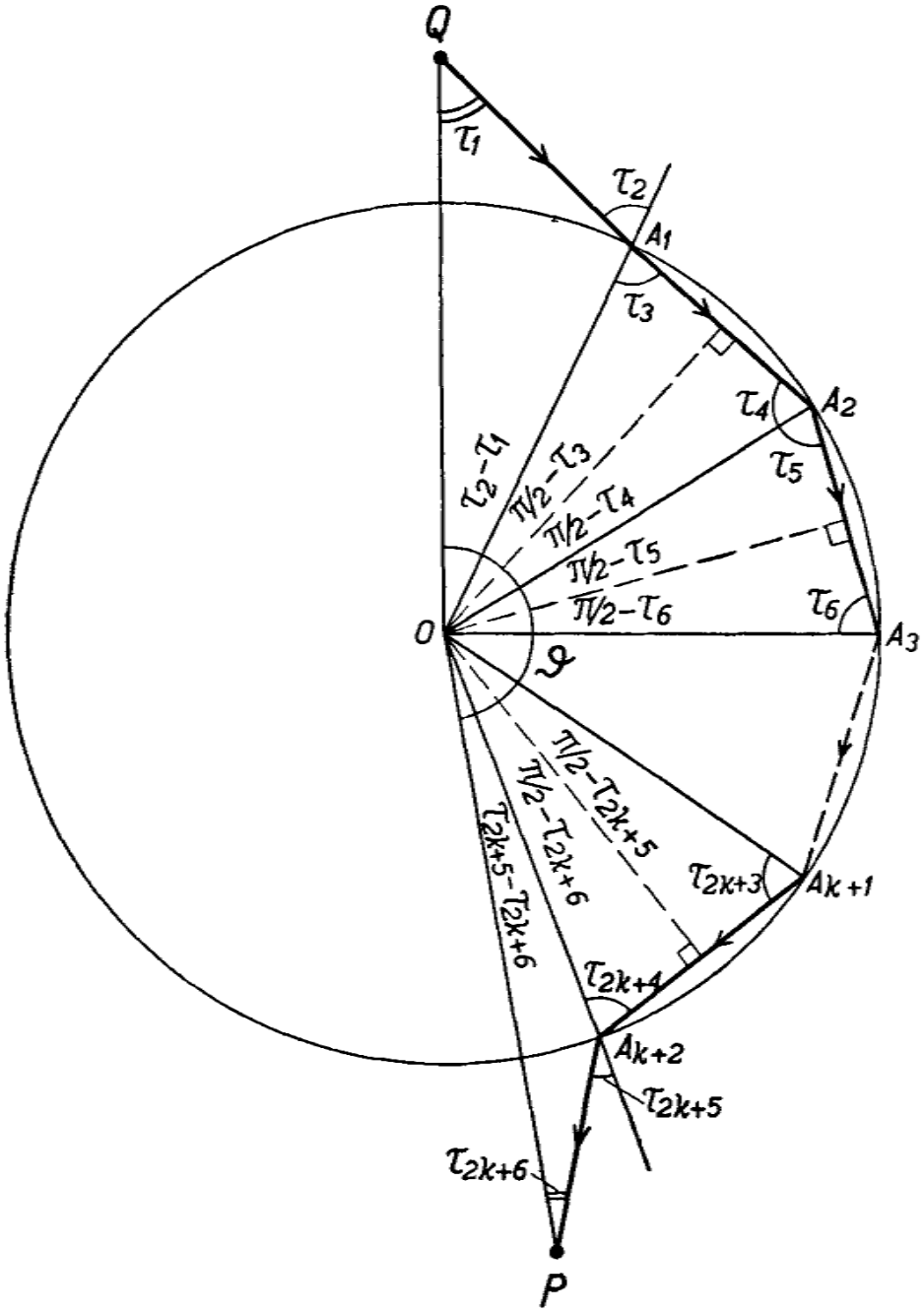
can often be approximated, as already considered, by first finding the position of the saddle point z_0, determined by $f'(z_0)=0$. The path of integration may then often be chosen in one of two ways, determined by

(*a*) $Im\{f(z)\}=Im\{f(z_0)\}$, (steepest descend, Debye, Watson);

(*b*) $Re\{f(z)\}=Re\{f(z_0)\}$, (stationary phase, Kelvin).

In both cases the main contribution to the integral is furnished by the immediate neighbourhood of the saddle point z_0. In case (a) this follows from the fact

Fig. 13.



The geometrical optical ray joining Q and P.

that the *modulus* of the integrand most rapidly decreases as we leave z_0, while in case (b) the phase changes more and more rapidly as we leave z_0, causing a cancellation

of the contribution to the integral by the parts of the path far away from z_0 *.

Developing further in (82) $f(z)$ as

$$\int e^{f(z)}\,dz = \int dz\; e^{f(z_0)+\frac{(z-z_0)^2}{2!}f''(z_0)+\frac{(z-z_0)^3}{3!}f'''(z_0)+\frac{(z-z_0)^4}{4!}f^{\mathrm{IV}}(z_0)\ldots},$$

we may get a first, second, third, and fourth approximation to the integral by retaining of the exponential series one, two, three or four terms resp. only.

Integrals of the type (82) often occur in diffraction problems. When they correspond to two-dimensional or three-dimensional space, the latter with axial symmetry, $f(z)$ depends not only upon the complex variable z but also on two spatial coordinates x and y. In this case the *first* approximation diverges for all values of x and y when, as in our problem, the limits of integration extend towards infinity. However, this first approximation converges when the limits are finite, and is then usually called the Fraunhofer approximation.

The *second* approximation to (82) leads to *Fresnel integrals*, whose physical interpretation is beautifully illustrated by the well-known *Cornu-spirals*. However, this second approximation may also diverge, viz., in the case when $f''(z_0)=0$. This equation, considered as a function of x and y, determines a curve—the caustic, the envelope of the rays —where therefore this approximation tends to infinity, as will further be shown in the next paragraph.

Considering next the *third* approximation, which leads to Airy's integrals, having the form

$$\int e^{-iz^3+i\alpha z}\,dz,$$

we observe that they may diverge if simultaneously we have $f''(z_0)=f'''(z_0)=0$, being two equations in x and y, and therefore determining isolated points in space only. These points are the cusps of the caustic, *i. e.*, the foci.

Finally, the *fourth* approximation would only diverge if simultaneously we have $f''(z_0)=f'''(z_0)=f^{\mathrm{IV}}(z_0)=0$, which in physical problems would lead to three equations in the two variables x and y, which in general are contradictory.

* For a further treatment of these questions we refer to the important paper by L. Brillouin, *Ann. Ec. Norm.* (2) xxxiii., Jan. 1916. See also Van der Corput, 'Compositio mathematica,' i. pp. 15–38 (1934) and iii. pp. 328–372 (1936); J. Bijl, Dissertation, Groningen, 1937.

The multi-dimensional integrals (79) occurring in our problem are a more-dimensional extension of the integral of the type (82). Also in this case we can consider the first, second, third . . . approximations as defined above, and they also diverge in a smaller region of the space coordinates the higher the order of approximation. Therefore a higher order approximation is necessary the nearer the caustic we wish to investigate the field. Especially in the case of the rainbow, for which at infinity the minimum of deviation coincides with the caustic, and as appreciable light is only to be found near this minimum we are forced to consider the third approximation (see § 16). Also in the radio case, between the transmitter and its horizon—at which latter point we reach the caustic—our second approximation remains finite only to diverge at the horizon itself.

In the next paragraphs we shall consider in more detail the evaluation of the second and third approximations for the integrals of the form (79). We saw already that they represent rays, and our next task therefore is to find their intensities.

15. *The Second-Order Approximation to the Intensities of the Rays.*

Our integrals of the form (79) being of a multi-dimensional type, we first give a short *résumé* of their second-order approximation. We therefore consider

$$\mathrm{I}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{f(z_1,\,z_2,\,\ldots\,z_n)}\,dz_1,\,dz_2\ldots dz_n, \quad (83)$$

the saddle point $(z_1^0, z_2^0, \ldots z_n^0)$ of which is determined by

$$\frac{\partial f}{\partial z_1}=\frac{\partial f}{\partial z_2}=\ldots=\frac{\partial f}{\partial z_n}=0.$$

Now the second approximation of (83) is obtained with the aid of the more-dimensional Taylor development of $f(z_1, z_2 \ldots z_n)$ round the saddle point, stopping at the second-order terms, leading to

$$\mathrm{I}\sim\mathrm{I}_2=e^{f(z_1^0,\,z_2^0,\,\ldots\,z_n^0)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \times e^{\frac{1}{2}(z_1-z_1^0)^2\left(\frac{\partial^2 f}{\partial z_1^2}\right)_0+(z_1-z_1^0)(z_2-z_2^0)\cdot\left(\frac{\partial^2 f}{\partial z_1\partial z_2}\right)_0+\cdots} dz_1\,dz_2\ldots dz_n. \quad (84)$$

Further, the exponent of (84) may be transformed to " normal coordinates " with the aid of the linear transformation

$$z_i - z_i^0 = \sum_{k=1}^{n} p_{ik} u_k,$$

so that the exponent becomes $\sum_{k=1}^{n} a_k u_k^2$, and the multiple integral (84) reduces to the product of n complete error integrals. The result is

$$I_2 = \frac{(i\sqrt{\pi})^n . |p_{ik}|}{\sqrt{a_1 a_2 \ldots a_n}} . e^{f(z_1^0, z_2^0, \ldots z_n^0)}, \quad . \quad . \quad (85)$$

where $|p_{ik}|$ means the determinant of the coefficients p_{ik}. Moreover, according to a well-known theorem of linear transformations we have

$$\sqrt{a_1 a_2 \ldots a_n} = |p_{ik}| . \sqrt{\frac{\Delta}{2^n}},$$

where Δ is Hesse's determinant of order n of $f(z)$, tken at the saddle point, and which is defined by

$$\Delta = \left| \frac{\partial^2 f}{\partial z_i \partial z_k} \right|_0 .$$

Hence (85) leads to the final result

$$I_2 = \frac{(i\sqrt{2\pi})^n}{\sqrt{\Delta}} . e^{f(z_1^0, z_2^0, \ldots z_n^0)}, \quad . \quad . \quad . \quad (86)$$

where, as we shall see further on, the first factor gives the amplitude and the last the phase in our ray problem.

We are now ready to apply (86) to (79), in order to find the second approximation of (79). We therefore identify $f(z_1, z_2 \ldots z_n)$ of (86) with $f^e_{K,m}$ as given by (80), and this for the values of the z's in the saddle point which are known from (81). Referring to fig. 13 we can express the result as

$$(f^e_{K,m})_0 = i\{k_1 . \overline{QA_1} + k_2(\overline{A_1A_2} + \ldots + \overline{A_{K+1}A_{K+2}}) + k_1 . \overline{A_{K+2}P}\} = iS_K, \text{ say}, \quad (87)$$

where S_K is the "*eiconal*" * belonging to the ray which has been reflected internally K times, and which expresses the phase of the ray. In order to find its amplitude we must evaluate Δ, and a further consideration shows that Δ is most elegantly expressed with the aid of

$$\frac{\partial \vartheta}{\partial n} = -\frac{1}{k_1 b \cos \tau_1} + \frac{1}{k_1 a \cos \tau_2} - \sum_{\nu=3}^{2K+4} \frac{1}{k_2 a \cos \tau_\nu} + \frac{1}{k_1 a \cos \tau_{2K+5}} - \frac{1}{k_1 r \cos \tau_{2K+6}}, \quad (88)$$

as

$$\Delta = (-1)^K . k_1 b . k_1 a . (k_2 a)^{2(K+1)} . k_1 a . k_1 r . (n - \tfrac{1}{2}) \sin \vartheta e^{-i\vartheta} . \cos \tau_1 \cos \tau_2 \ldots \cos \tau_{2K+6} \cdot \frac{\partial \vartheta}{\partial n}.$$

The geometrical meaning of $\frac{\partial \vartheta}{\partial n}$ is obtained if, by keeping the transmitter Q (fig. 13) fixed, we imagine the receiver P to move a little distance along its circle round O. Thereby ϑ and n vary with a differential quotient $\frac{\partial \vartheta}{\partial n}$. It is further of interest to note here that the physical geometrical meaning of the variable n, which originally appeared as a separation constant in the harmonic series (1) from which our whole investigation started, according to (81) is nothing but 2π times the length of the orthogonal from the centre O on all cords of the broken ray (expressed in wave-lengths), and which is even true for those parts of the ray which lie outside the sphere.

Returning to (79) $A(n)$ has now to be taken at the saddle point, and further introducing in $A(n)$ approximations of the type

$$\zeta^{(1)}_{ka \sin \tau} \cdot \zeta^{(2)}_{ka \sin \tau} \sim \frac{1}{(ka)^2 \cos \tau}, \quad . \quad . \quad . \quad (89)$$

and limiting ourselves to $m=0$, we find for $O^e_{K,0}$ the final result (second-order approximation)

$$O^e_{K,0} \sim \frac{i^K . R_{12} R^K_{22} R_{21}}{k_1 b . k_1 r \sqrt{\sin \vartheta . \frac{\partial \vartheta}{\partial n}}} \cdot \frac{\sqrt{k_1 a \sin \tau_2}}{\sqrt{\cos \tau_1 . \cos \tau_{.K+6}}} . e^{iS_K}. \quad (90)$$

* See also. Ch. Manneback, 'Recueil des travaux de l'Assemblée Générale de l'Union Radio Scientifique Internationale, tenue à Londres en Sept. 1934': Bruxelles, 1935.

This formula, which represents the ray of order K, confirms the considerations given in § 14 in so far as it becomes infinite where $\frac{\partial \vartheta}{\partial n}=0$, *i. e.*, on the caustic.

With a view to the radio application, where the ray of order -1, which is the ray which has been reflected only once externally and which has to be computed separately, we give here its approximate expression in the form

$$O^{e}_{-1,0} \sim \frac{\sqrt{k_1 a \sin \tau_2}.R_{11}.e^{iS-1}}{ik_1 b.k_1 r\sqrt{\sin \vartheta . \frac{\partial \vartheta}{\partial n}.\cos \tau_1.\cos \tau_4}}$$

$$=\alpha_{11}.R_{11}.\frac{e^{ik_1(R_1+R_2)}}{ik_1(R_1+R_2)}, \quad . \quad (91)$$

where

$$\alpha_{11}=\frac{(R_1+R_2)\sqrt{k_1 a \sin \tau_2}}{k_1 br\sqrt{\sin \vartheta . \frac{\partial \vartheta}{\partial n}.\cos \tau_1 \cos \tau_4}},$$

which, after a slight transformation, becomes

$$\alpha_{11}=\frac{a(R_1+R_2)\sqrt{\sin \tau_2.\cos \tau_2}}{\sqrt{br}\sin \vartheta(R_1 r\cos \tau_4+R_2 b\cos \tau_1)}. \quad . \quad (92)$$

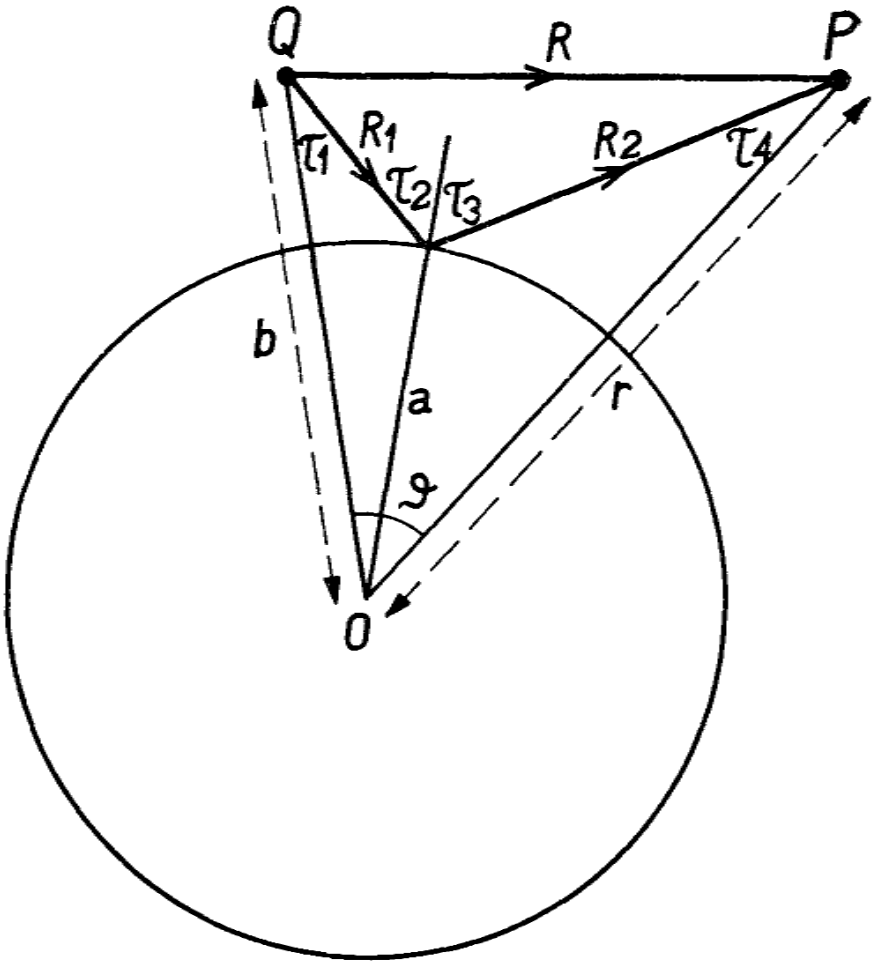
Summarizing, we refer to fig. 14 for the meaning of the variables τ and R.

Formula (91) tells us the interesting physical fact that the reflected ray is *not* given by the simple product of the primary ray and the reflexion coefficient R_{11}, but that a factor α_{11}, which we call the " divergence factor," also enters. This is physically obvious, because the reflexion in our case occurs against a curved surface, so that the divergence of two nearby primary rays, contrary to what happens for a reflexion against a plane boundary, is increased, and therefore the amplitude of the secondary ray is diminished, due to the increased divergence (astigmatism). This is confirmed in the two special cases (*a*) $r=a$ (receiver on the sphere), where this extra divergence obviously disappears, and we therefore obtain $\alpha_{11}=1$, and (*b*) $a\to\infty$ (reflexion against plane), where, again, $\alpha_{11}=1$.

Applying, finally, our second approximation to $\mathrm{I}^{e}_{-1,0}$ (of (70)), we find

$$\mathrm{I}^{e}_{-1,0} \sim \Pi_{pr} = \frac{e^{ik_1 \mathrm{R}}}{ik_1 \mathrm{R}},$$

Fig. 14.



The direct ray together with the once externally reflected ray.

so that in the radio case the second approximation to the total field becomes

$$\Pi_{\mathrm{tot}} \sim \mathrm{I}^{e}_{-1,0} + \mathrm{O}^{e}_{-1,0} \sim \frac{e^{ik_1 \mathrm{R}}}{ik_1 \mathrm{R}} \{1 + \alpha_{11} R_{11} e^{ik_1(\mathrm{R}_1 + \mathrm{R}_2 - \mathrm{R})}\}. \quad (93)$$

This extended reflexion formula differs from one occasionally found in the technical literature, where the divergence factor α_{11} is taken as unity, and instead of our spherical reflexion coefficient R_{11} the ordinary plane reflexion

coefficient is used. However, the numerical difference is small in most radio applications.

16. *The Third-Order Approximation to the Intensities of the Rays.*

Having in the previous paragraph considered the second approximation of the elementary waves $O^{e}_{K,0}$ as given by (79), we shall now investigate the third approximation of these waves. As pointed out already in §14, the third approximation is the fundamental one in the theory of the rainbow, because here light is practically only present near the minimum of deviation, which coincides with the caustic at infinity.

Developing the exponent (80) round the saddle point up to the third-order terms, it is found that the variables τ occur separated, so that the multiple integration over all the τ's can be affected first, thus leading, with $\tau_V - \tau_V{}^0 = u$, to a product of integrals of the type

$$\int_{\substack{w_1\\ w_2}} e^{-i\alpha u^3 \mp i\beta u^2 - i\gamma u} du = \frac{\pi}{3\sqrt{\alpha}} \cdot e^{\pm i\frac{\pi}{6} - \frac{2}{27} i \frac{\beta^3}{\alpha^2} \pm \frac{i\beta\gamma}{3\alpha}}$$

$$\cdot H^{(1)}_{(2)1/3}\left\{\frac{2\left(\frac{\beta^2}{3\alpha} - \gamma\right)^{3/2}}{3^{3/2}\sqrt{\alpha}}\right\} \sim \sqrt{\frac{\pi}{\beta}} \cdot e^{\mp i\frac{\pi}{4} \pm \frac{i}{4}\frac{\gamma^2}{\beta} \pm \frac{i}{8}\frac{\alpha\gamma^3}{\beta^3}}, \qquad (94)$$

where the paths of integration are as given in fig. 15, W_1 belonging to the functions $\zeta^{(1)}$ and W_2 to the functions $\zeta^{(2)}$.

The last approximation of (94) can be obtained in two different ways : (*a*) as the first term of the asymptotic series of the Hankel function, and (*b*) by applying (86) to the integral of (94). In both cases it is assumed that

$$\beta^3 >> \alpha^2,$$

$$\beta^2 >> \alpha\gamma,$$

which in the case of (79) comes to

$$ka \cdot \frac{\cos^3\tau}{\sin^2\tau} >> 1,$$

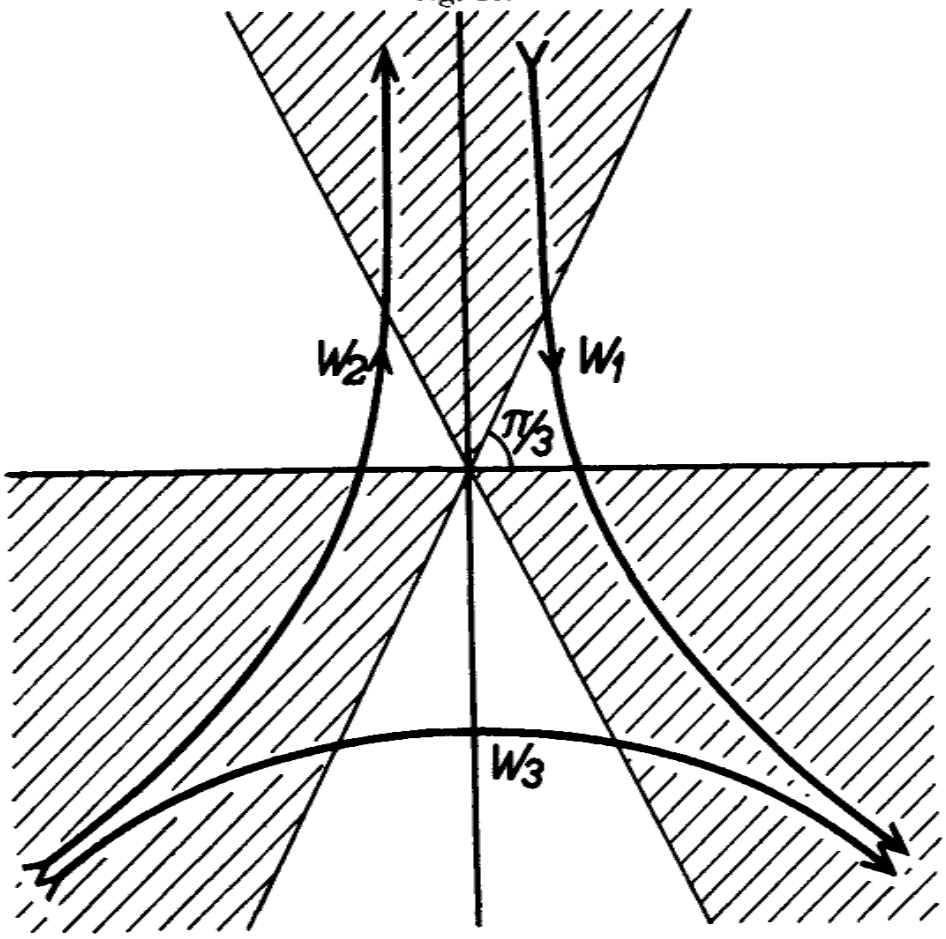
$$ka \cdot \frac{\cos^2\tau}{\sin\tau} >> n - n_0.$$

Having thus affected all the τ integrations we are left with the integrals over ϕ and n. The integral over ϕ

appears to be a nearly complete Fresnel integral, which can be approximated by unity. The remaining integral over n contains an exponent which can be developed again up to the third-order terms in $n-n_0$, thus leading to an integral of the form (*cf.* fig. 15) :

$$\int_{w_3} e^{-i(n-n_0)^3+i\delta(n-n_0)}\,d(n-n_0)=\frac{\pi}{3}\sqrt{\delta}e^{i\pi/6}.\mathrm{H}^{(1)}_{1/3}\left(\frac{2\delta^{3/2}}{3^{3/2}}.e^{-i\pi}\right),$$

Fig. 15.



Path of integration of third-order integrals.

giving as final result

$$\mathrm{O}^{e}_{\mathrm{K},0}\sim\frac{(-1)^{\mathrm{K}+1}e^{\frac{5}{12}i\pi+i\mathrm{S}_{\mathrm{K}}}\sqrt{\pi a\sin\tau_2}}{k_1^{3/2}br\sqrt{6}\sin\vartheta.\cos\tau_2.\cos\tau_{2\mathrm{K}+6}}.\mathrm{R}_{12}\mathrm{R}^{\mathrm{K}}_{22}\mathrm{R}_{21}$$
$$.\frac{\frac{\partial\vartheta}{\partial n}}{\frac{\partial^2\vartheta}{\partial n^2}}.e^{-\frac{1}{3}\left(\frac{\partial\vartheta}{\partial n}\right)^3/\left(\frac{\partial^2\vartheta}{\partial n^2}\right)^2}.\mathrm{H}^{(1)}_{1/3}\left\{\frac{\left(\frac{\partial\vartheta}{\partial n}\right)^3}{3\left(\frac{\partial^2\vartheta}{\partial n^2}\right)^2}\right\},\qquad(95)$$

where in the result analogous to (88) we have, apart from $\frac{\partial \vartheta}{\partial n}$, also introduced $\frac{\partial^2 \vartheta}{\partial n^2}$, the latter being given by the geometrical relation

$$\frac{\partial^2 \vartheta}{\partial n^2} = -\frac{\sin \tau_1}{k_1^2 b^2 \cos^3 \tau_1} + \frac{\sin \tau_2}{k_1^2 a^2 \cos^3 \tau_2} - \sum_{\nu=3}^{2K+4} \frac{\sin \tau_\nu}{k_2^2 a^2 \cos^3 \tau_\nu} + \frac{\sin \tau_{2K+5}}{k_1^2 a^2 \cos^3 \tau_{2K+5}} - \frac{\sin \tau_{2K+6}}{k_1^2 r^2 \cos^3 \tau_{2K+6}}; \quad (96)$$

(95) therefore constitutes our third approximation to the elementary waves represented by (79).

The factor $H_{1/3}^{(1)}$ contains in the argument the factor $\left(\frac{\partial \vartheta}{\partial n}\right)^3$, which is zero on the caustic and has different signs on its sides, and this nicely demonstrates a well-known general property of waves near caustics, viz., that their amplitude falls off exponentially at one side of the caustic and in an oscillatory manner at the other side, this fact being demonstrated by the exponential behaviour of $H_{1/3}^{(1)}(x)$ for $x>0$ and the oscillator character of the same function for $x<0$. This behaviour is due to the fact that at the convex side of the caustic, where $\frac{\partial \vartheta}{\partial n}<0$, each point is in general reached by two different rays causing an interference pattern.

It is also possible, as a further analysis shows, to give the geometrical meaning of the argument of $H_{1/3}$ in (95). In fig. 16 the caustic is represented by the full-drawn curve C, while the wave-front W, cutting the caustic C in the cusp W_0 orthogonally, is shown dotted *. Further, there are represented two rays, W_1P and W_2P, touching the caustic at K_1 and K_2 resp., and which rays arrive at P with a path difference $\triangle = \overline{W_1P} - \overline{W_2P}$. Due to the caustic being the evolute of the wave-front W, this expression may be replaced by

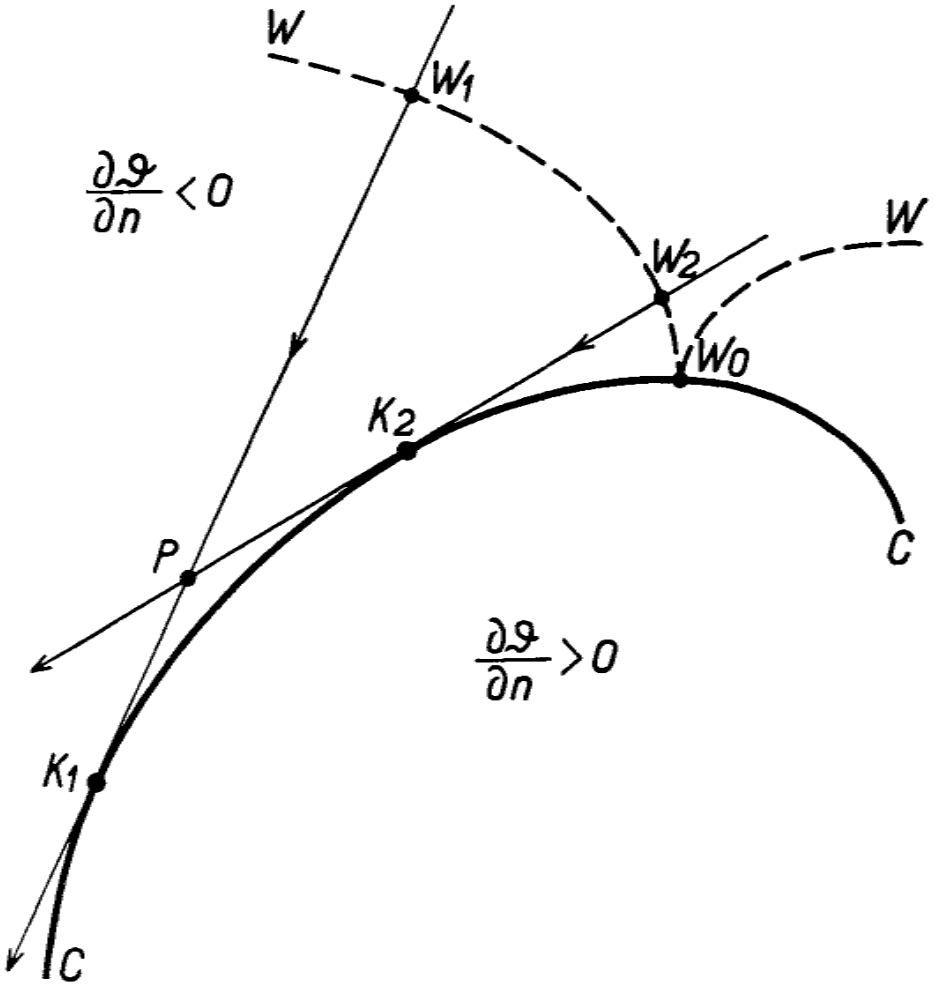
$$\Delta = \overline{K_1P} + \overline{PK_2} - \text{arc } K_1K_2,$$

* See also R. Wood, ' Physical Optics,' p. 148 (New York, 1919).

and a somewhat elaborate transformation, moreover, shows that

$$-\frac{2}{3}\frac{\left(\frac{\partial\vartheta}{\partial n}\right)^3}{\left(\frac{\partial^2\vartheta}{\partial n^2}\right)^2}\sim k_1\Delta,$$

Fig. 16.



Caustic with wave-front.

so that the argument of $H_{1/3}$ in (95) simply becomes $\frac{1}{2}.e^{-i\pi}.k_1\triangle$. The asymptotic expression of $H_{1/3}$ then yields two terms, representing the two rays with a phase difference $k_1\Delta$. We will return to these third-order approximations in § 18.

17. *Numerical Results for the Radio Case.*

Summarizing, we have now available for numerical interpretation for the radio case the following expressions :—

(*a*) Watson's expression, which is a special case of (15), viz. for $b=a$, and therefore only refers to the case where both transmitter and receiver are situated on the ground, and which is based on the residue method.

(*b*) The expression (15), which is also based on the residue method, but here either the transmitter or the receiver is situated above the ground *.

(*c*) The expression (77), covering all cases, and which was derived with the residue method after the total field was broken up into elementary waves and after it was shown that in the radio case waves reflected against the centre of the sphere were negligible. Thus the denominator $\frac{\partial M}{\partial n}$ of (77) differs from Watson's $\frac{\partial N}{\partial n}$ of (15) as a consequence of the fact that the direct radiation and the only once externally reflected radiation were maintained.

(*d*) The reflexion formula (93), which was derived after the total field was broken up into elementary waves, the direct radiation and the only once externally reflected radiation being maintained, the derivation being obtained with the aid of the saddle point method, yielding (1) the spherical reflexion coefficient R_{11}, and (2) the divergence factor α_{11}.

As to the ranges of applicability of the different expressions, these are most clearly described when we consider first the field far beyond the horizon of the transmitter. There formula (77) can be used, and its first term already gives a very good approximation. When we approach the horizon from beyond the number of terms of (77) to be considered rapidly increases, although at the horizon no more than six to ten terms are necessary in practice. Going further towards the transmitter, as we have passed the horizon, the number of terms of (77) to be retained rapidly increases still further, and in our examples we have gone up to eighteen terms.

* After the present investigation was finished a paper appeared by Wwedensky, ' Technical Physics of U.S.S.R.', iii. p. 915 (1936), giving a formula analogous to our expression (15).

This shows the practical applicability of our general expression (77), which, however, is formally valid everywhere even when the receiver approaches the transmitter indefinitely.

We have further found it convenient in applying (77) to limit ourselves to two limiting cases (which have already been considered before), viz.,

(1) *Total reflexion* : $\sigma=\infty$ (see § 7 *b*),

(2) *Maximum absorption* :

$$|\delta| = \left| \frac{i}{x^{1/3}} \cdot \frac{k_2^2/k_1^2}{\sqrt{k_2^2/k_1^2-1}} \right| << 1$$

(see § 8),

for in these cases the values of τ_s and τ_s' can easily be calculated (see (40 *a*) and (55)).

It is thus seen that, while approaching the transmitter, the number of necessary terms of (77) soon becomes unwieldy. Mostly there is still left a transition region, dotted in the figures to be given later on, but then we soon approach shorter distances from the transmitter, where we can apply the reflexion formula (93). The values of the field in the transition region can, however, mostly be properly estimated by interpolation.

Already in Part I. of this paper we gave some numerical results also for either the transmitter or the receiver above the ground, these cases being equivalent according to the principle of reciprocity (see fig. 8). The effect in (77) of raising the height is expressed by the factors

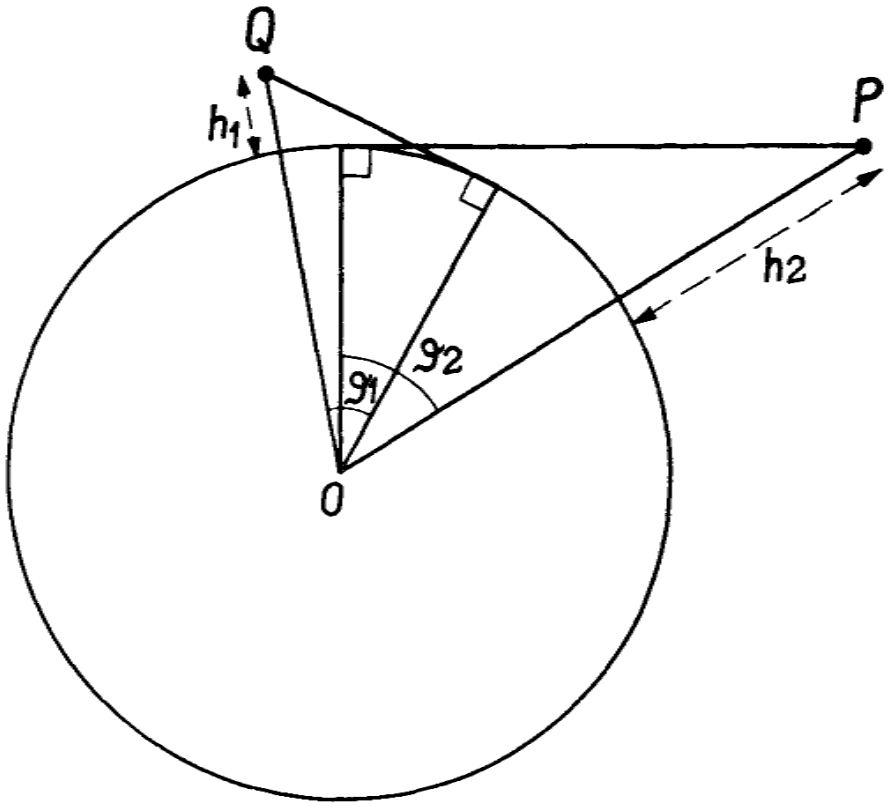
$$f_s = \left\{ \frac{\zeta_n^{(1)}(k_1 b)}{\zeta_n^{(1)}(k_1 a)} \right\}_{n=x+x^{1/3}\tau_s},$$

present in each term of the series. If both transmitter and receiver are raised above the ground two of these factors are present, the second one with b replaced by r. In order to discuss the numerical effect of these factors f_s the ζ functions can be replaced by the approximation (17), yielding (after a slight simplification based on $x^{1/3}\vartheta_0^5 << 1$, where $\vartheta_0 = \cos^{-1} b/a$) :

$$f_s \sim \sqrt{1-\frac{\chi_0^2}{2\tau_s}} \cdot \frac{H_{1/3}^{(1)}\left\{\frac{\chi_0^3}{3}\left(1-\frac{2\tau_s}{\chi_0^2}\right)^{3/2}\right\}}{H_{1/3}^{(1)}\left\{\frac{(-2\tau_s)^{3/2}}{3}\right\}}, \quad . \quad (97)$$

and a similar expression when τ_s is replaced by τ_s'. Here we have introduced the new variable $\chi = x^{1/3}\,\vartheta = (ka)^{1/3}$; the special value of χ for a point at the horizon is designated by $\chi_0 = x^{1/3}\vartheta_0$ if either the transmitter or the receiver is above the ground. If both are raised above the ground (see fig. 17) the transmitter as well as the receiver each has its own horizon, for which the χ's are denoted by $\chi_1 = x^{1/3}\vartheta_1$ and $\chi_2 = x^{1/3}\vartheta_2$.

Fig. 17.



Definitions of ϑ_1 and ϑ_2.

Returning to (97) a very important physical property is expressed by these factors, viz., the increase of the field with the height. This increase appears to be a function of (*a*) the numerical constants τ_s depending upon the electrical constants of the ground and the wave-length and (*b*) the geometrical variable χ_0. Now it appears that the argument of the Hankel function in the nominator of (97) is of the order $1/3\,\chi_0^3$, and the numerical behaviour of this Hankel function is totally different in the two cases $1/3\,\chi_0^3 < 1$ and $1/3\,\chi_0^3 > 1$, corresponding

resp. to $h<56\lambda^{2/3}$ and $h>56\lambda^{2/3}$, where $h=b-a$ and λ are both measured in metres *.

When $h<<56\lambda^{2/3}$ the Hankel functions in (97) may be approximated by the first terms of their power series development, leading to

$$f_s\sim\frac{1+0\cdot459e^{2/3i\pi}(\chi_0{}^2-2\tau_s)-0\cdot0417(\chi_0{}^2-2\tau_s)^3\ldots}{1+0\cdot459e^{2/3i\pi}(-2\tau_s)-0\cdot0417(\chi_0{}^2-2\tau_s)^3\ldots}. \quad (98)$$

When, on the other hand, $h>>56\lambda^{2/3}$, we approximate the Hankel function in the nominator by the first term of the asymptotic development, while the Hankel function in the denominator must first be written as

$$H^{(1)}_{1/3}(z)=\frac{e^{-i\pi/6}}{\sin\pi/3}\cdot\{J_{1/3}(ze^{i\pi})+J_{-1/3}(ze^{i\pi})\},$$

and then the first term of the asymptotic development of both $J_{1/3}$ and $J_{-1/3}$ can be taken. This leads to the following final formulæ, valid therefore for $\chi_0{}^3>>3$, $\chi_1{}^3>>3$, $\chi_2{}^3>>3$:

Values of τ :

(1) $\sigma=\infty$: $\quad\tau_s\sim\frac{1}{2}\{3\pi(s+\frac{1}{4})\}^{2/3}\cdot e^{i\pi/3}$,

(2) $|\delta|<<1$: $\quad\tau_s'\sim\frac{1}{2}\{3\pi(s+\frac{3}{4})\}^{2/3}\cdot e^{i\pi/3}-\delta$,

$$\left(\text{for } s<<\frac{1}{30\delta^3}\right).$$

Values of height-gain factors :

$$\left.\begin{aligned}&(1)\ \sigma=\infty:\\&f_s\sim\frac{(-1)^s(-2\tau_s)^{1/4}}{2\sqrt{\chi_0}}\cdot e^{i\frac{\chi_0{}^3}{3}\left(1-\frac{2\tau_s}{\chi_0{}^2}\right)^{3/2}+i\frac{\pi}{4}},\\&(2)\ |\delta|<<1:\\&f_s\sim-\frac{12}{5}\frac{(-1)^s(-2\tau_s')^{7/4}}{\sqrt{\chi_0}}\cdot\frac{e^{i\frac{\chi_0{}^3}{3}\left(1-\frac{2\tau_s'}{\chi_0{}^2}\right)^{3/2}+i\frac{\pi}{4}}}{\left\{1-\frac{96}{5}\delta\tau_s'{}^2\right\}}.\end{aligned}\right\} \quad (99)$$

* In an investigation of the same problem by Mr. T. L. Eckersley, who recently kindly communicated to us his results (not yet published) obtained with his " phase integral method," the effect of raising the transmitter is called the " height gain " ; he also finds a critical height given as $h=50\lambda^{2/3}$, which corresponds well with our $h=56\lambda^{2/3}$, as found above.

Attenuation factors for $b=r=a$:

$$\left.\begin{aligned} &(1)\ \sigma=\infty: \quad \frac{\Pi_{\text{tot}}}{\Pi_{\text{pr}}}\sim\sqrt{2\pi i\chi}\cdot\sum\frac{e^{i\tau_s\chi}}{\tau_s},\\ &(2)\ |\delta|<<1: \ \frac{\Pi_{\text{tot}}}{\Pi_{\text{pr}}}\sim-2\delta^2\cdot\sqrt{2\pi i\chi}\cdot\sum e^{i\tau_{s'}\chi}. \end{aligned}\right\}\quad . \quad . \quad (100)$$

Attenuation factors for $b\neq a$, $r=a$:

$$\left.\begin{aligned} &(1)\ \sigma=\infty: \\ &\frac{\Pi_{\text{tot}}}{\Pi_{\text{pr}}}\sim\sqrt{-\frac{2\pi\chi}{\chi_0}}\cdot\sum\frac{(-1)^{s+1}}{(-2\tau_s)^{3/4}}\cdot e^{i\tau_s\chi+i\frac{\chi_0^3}{3}\left(1-\frac{2\tau_s}{\chi_0^2}\right)^{3/2}},\\ &(2)\ |\delta|<<1: \\ &\frac{\Pi_{\text{tot}}}{\Pi_{\text{pr}}}\sim\frac{24}{5}\delta^2\cdot\sqrt{-\frac{2\pi\chi}{\chi_0}}\cdot\sum(-1)^s(-2\tau_s')^{7/4}\\ &\qquad\cdot\frac{e^{i\tau_s'\chi+\frac{i\chi_0^3}{3}\left(1-\frac{2\tau_s'}{\chi_0^2}\right)^{3/2}}}{\left\{1-\frac{96}{5}\delta\tau_s'^2\right\}}. \end{aligned}\right\}\quad . \quad (101)$$

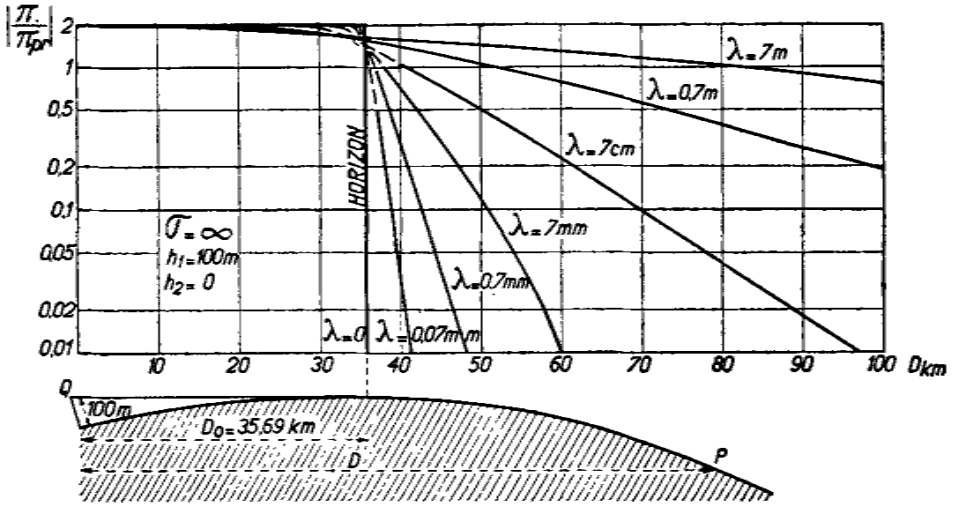
Attenuation factors for $b\neq a$, $a=r$:

$$\left.\begin{aligned} &(1)\ \sigma=\infty: \\ &\frac{\Pi_{\text{tot}}}{\Pi_{\text{pr}}}\sim\sqrt{\frac{\pi\chi}{2i\chi_1\chi_2}}\\ &\qquad\cdot\sum\frac{e^{i\tau_s\chi+\frac{i\chi_1^3}{3}\left(1-\frac{2\tau_s}{\chi_1^2}\right)^{3/2}+\frac{i\chi_2^3}{3}\left(-\frac{2\tau_s}{\chi_2^2}\right)^{3/2}}}{\sqrt{-2\tau_s}},\\ &(2)\ |\delta|<<1: \\ &\frac{\Pi_{\text{tot}}}{\Pi_{\text{pr}}}\sim\frac{576}{25}\delta^2\cdot\sqrt{\frac{\pi\chi}{2i\chi_1\chi_2}}\\ &\cdot\sum\frac{(-2\tau_s')^{7/2}\cdot e^{i\tau_s'\chi+\frac{i\chi_1^3}{3}\left(1-\frac{2\tau_s'}{\chi_1^2}\right)^{3/2}+i\frac{\chi_2^3}{3}\left(1-\frac{2\tau_s'}{\chi_2^2}\right)^{3/2}}}{\left\{1-\frac{96}{5}\delta\tau'^{22}\right\}^2}. \end{aligned}\right\}\quad . \quad (102)$$

Figs. 18–20 were calculated with the aid of the above formulæ, where occasionally a still higher approximation for f_s than given above was made use of. These figures refer to

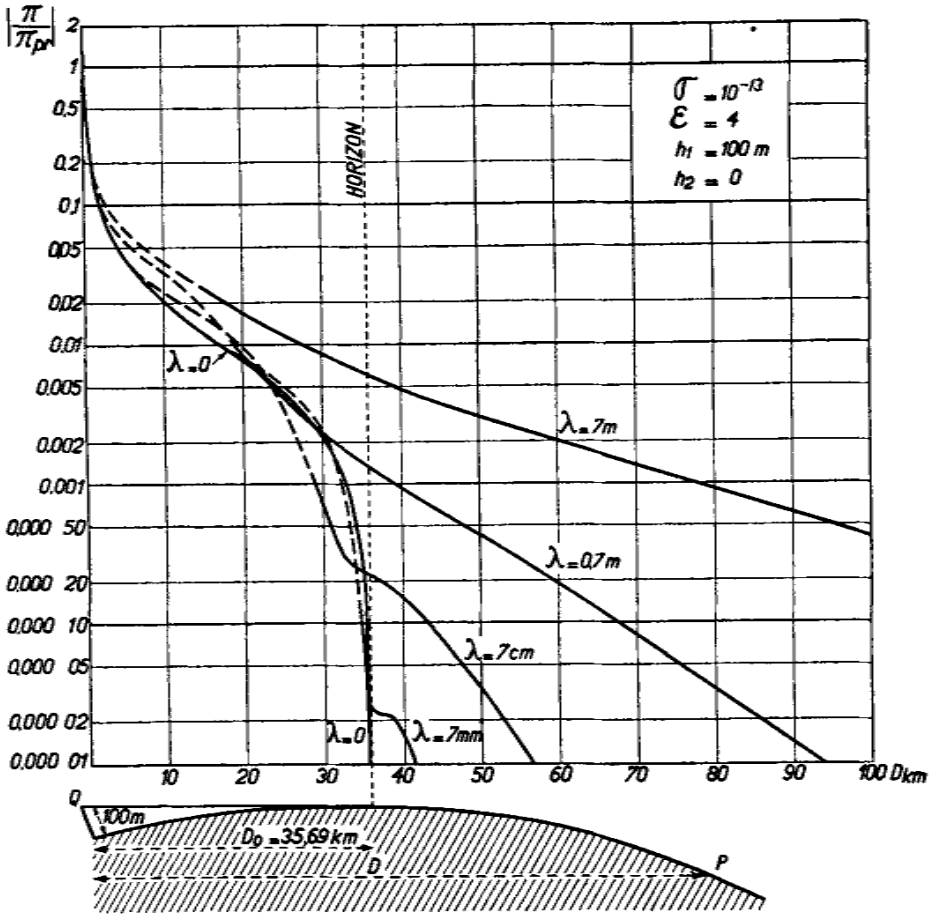
Fig. 18, $\sigma=\infty$, $h_1=100$ m., $h_2=0$,
Fig. 19, $\sigma=10^{-13}$, $\epsilon=4$, $h_1=100$ m., $h_2=0$,
Fig. 20, $\sigma=10^{-13}$, $\epsilon=4$, $h_1=h_2=100$ m.,

Fig. 18.



Influence of the wave-length on the field directly on the surface of the earth for a transmitter at a height of 100 m. ($\sigma = \infty$).

Fig. 19.

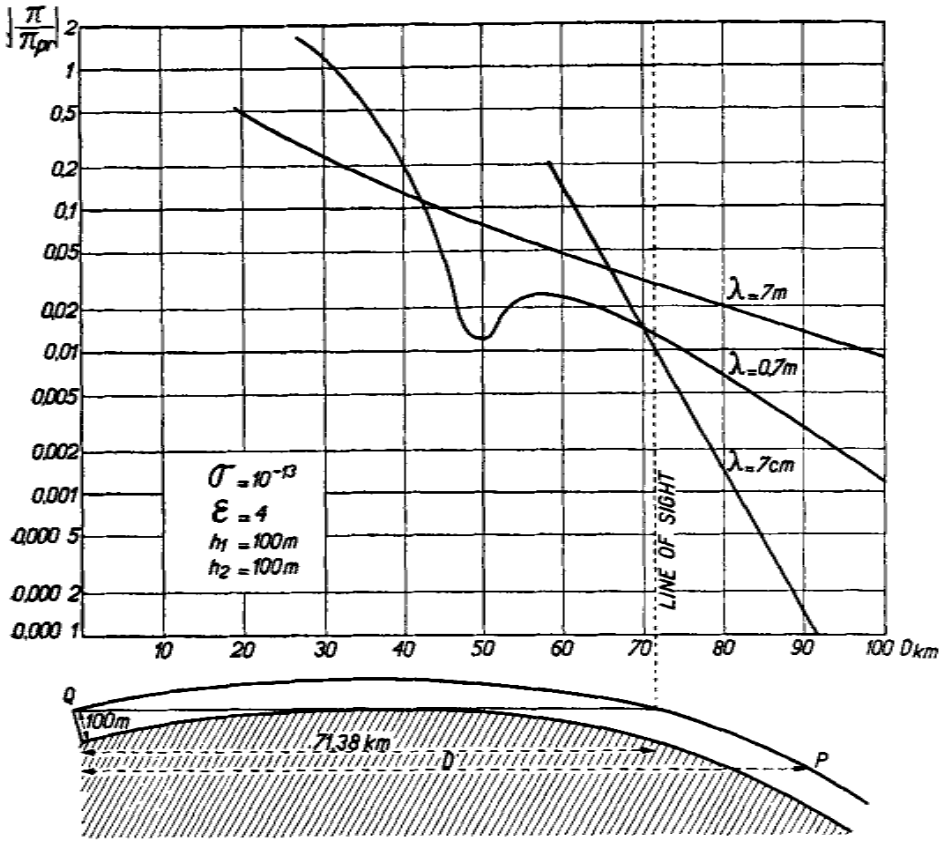


Influence of the wave-length on the field directly on the surface of the earth for a transmitter at a height of 100 m. ($\epsilon = 4$; $\sigma = 10^{-13}$).

and they represent $|\Pi_{tot}|$ divided by the modulus of the primary field $e^{ik_1R}/(ik_1R)$, and therefore represent the attenuation factors with which the undisturbed radiation must be multiplied due to the presence of the earth.

Fig. 18 (for $\sigma=\infty$) clearly shows that waves longer than about $\lambda=10$ cm. do not present a pronounced effect at the horizon, while shorter waves rapidly fall

Fig. 20.



Influence of the wave-length on the field for both transmitter and receiver at a height of 100 m. ($\epsilon=4$; $\sigma=10^{-13}$).

off behind the horizon, tending to a discontinuous drop to zero for waves $\lambda\to 0$.

Fig. 19 (with absorption) shows a similar behaviour of the waves, apart then from the absorption which is already present for all wave-lengths considered. The case of the limiting wave-length $\lambda\to 0$ is here fully determined by ϵ only, the conduction current becoming negligible in comparison with the displacement current.

Fig. 20 (with absorption) again shows a similar behaviour

of the waves ; only the wave $\lambda=70$ cm. has a pronounced minimum before the line of sight. The interference maxima and minima, which undoubtedly are present, fall outside the part of the curves so far calculated.

Herewith we conclude our investigation of the radio problem, and in the next paragraph we consider in further detail the application of our formulæ to the theory of the rainbow.

18. *Theory of the Rainbow.*

While in the radio case it was sufficient to consider the direct wave together with the wave which was once externally reflected only, because of the big absorption of all the other waves entering the earth, in considering the rainbow the conditions are just reversed, because the rainbow is observed near the minimum of deviation of rays which have been once ($K=1$; primary bow) and twice ($K=2$; secondary bow) reflected inside the raindrops.

But as our theory has been developed generally, we are ready to apply it in the present case. It only needs two extensions : (*a*) the source (here the sun) will be considered at infinite distance, and (*b*) the receiver (the observer) also at a very great distance. Moreover, if in the radio case we consider a vertical electric dipole at an infinite height above the earth, due to the radiation characteristic of this dipole being zero in its own axis, no energy would reach the earth at all. We now have therefore, following Debye*, to replace the electric dipole of the radio case by a combination of a contribution of an electric and a magnetic character, the sum of which represents a plane polarized wave.

Whereas in the classical theory of the rainbow the usual procedure was first to find the wave-front after K internal reflexions, then approximating this wave-front round its inflexion point and summing the different contributions from there with the aid of a Kirchhoff integral, leading in its turn to Airy's theory (the so-called Rainbow-integral), we will here follow a totally different way, and develop the result starting from our rigorous

* P. Debye, " Der Lichtdruck auf Kugeln von beliebigem Material " (Inaugural Dissertation, München, 1908); *Ann. d. Phys.* xxx. p. 57 (1909).

solution of the wave equation, which was split up in the subsidiary waves O^{e}_{K} of (70 *d*). The third-order approximation of the multi-dimensional saddle point integral of O^{e}_{K}, as considered in § 16, then leads to an extension of Airy's classical results. It is just this extension which here enables us to determine in a rigorous way (*a*) the absolute intensity of the rainbow, (*b*) the losses at the reflexions and refractions, and (*c*) the state of polarization. Whereas these points had formally to be introduced *ad hoc* in a more or less arbitrary way *, they are quite naturally present in our theory, due to the fact that it was derived from a rigorous wave solution.

In order to be able to compare later on our results with the classical ones we shall now recall some formulæ of the older theory, which gives us the opportunity to compare the notations used here with the classical ones.

Referring to fig. 21, let the sun be situated at the infinite point Q, corresponding to $\vartheta=0$, and the observer at the infinite point P at an angular distance ϑ from Q. A typical ray belonging to the primary bow ($K=1$) is represented by $QA_1A_2A_3P$, this ray having a total deviation $D=\pi-\vartheta$. As before, we will designate the rays by the value of the parameter n, see (81 *a*), which is given by

$$n=k_1a\sin\tau_2=k_2a\sin\tau_3=\ldots=k_2a\sin\tau_{2K+4}=k_1a\sin\tau_{2K+5}.$$

Expressed in the variable n, the deviation $D(n)$ then becomes

$$D(n)=2\sin^{-1}\left(\frac{n}{k_1a}\right)-2(K+1).\sin^{-1}\left(\frac{n}{k_2a}\right)+K\pi, \quad (103)$$

and the minimum of deviation, round which the light energy is concentrated, is given by

$$\frac{dD(n)}{dn}=0.$$

Further, for this ray, suffering the minimum of deviation, it is easily found that the corresponding τ_2 becomes $\tau_{2\,\min}$ determined by

$$\sin\tau_{2\,\min}=\sqrt{\frac{(K+1)^2-k_2^2/k_1^2}{K(K+2)}}, \quad . \quad . \quad (104)$$

* See, *e. g.*, W. Möbius, *Ann. d. Phys.* xxxiii. p. 1493 (1910), and xl. p. 736 (1913).

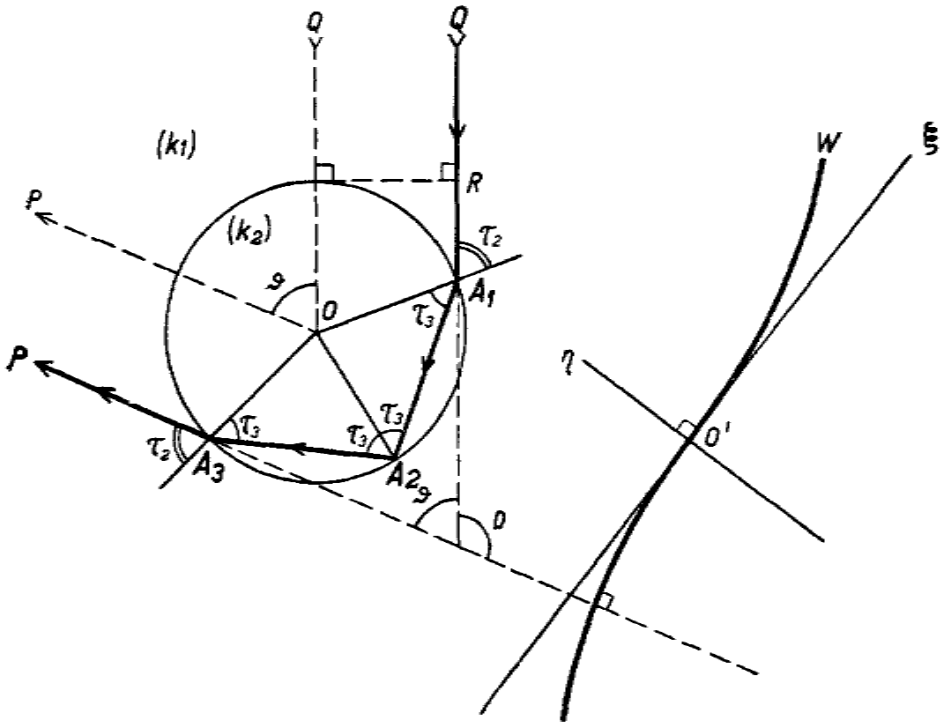
from which it follows that the ray of lowest order corresponds to K=1 (causing the primary bow), because for water in the optical range we have $k_2^2/k_1^2=1{\cdot}78$. The secondary bow corresponds with K=2.

Referring further to fig. 21, the classical theory considers the wave-front W determined by

$$k_1.\overline{RA_1}+k_2(\overline{A_1A_2}+\ldots+\overline{A_{K+1}A_{K+2}})+k_1.\overline{A_{K+2}P}=k_1a,$$

and the ray of minimum deviation cuts this wave-front at its inflexion point O′. It was further usual to introduce new rectangular coordinates ξ and η with O′ as centre.

Fig. 21.



The geometrical-optical ray QP in the primary rainbow, together with a wave-front W.

Expressed in these coordinates the first approximation for this wave-front round its inflexion point was given as

$$\eta=\frac{h}{3a^2}\cdot\xi^3,$$

with

$$h=\frac{K^2(K+2)^2}{(K+1)^2}\cdot\frac{\sqrt{(K+1)^2-k_2^2/k_1^2}}{(k_2^2/k_1^2-1)^{3/2}}, \quad . \quad (105)$$

and it was the interference of the waves originating from this wave-front round O′ which lead to the classical theory.

In the present electromagnetic theory of the rainbow we have to start with a plane polarized wave as primary field, *i. e.*

$$\mathrm{E}_x=-ik_1\,.\,e^{-ik_1z}\,;\quad \mathrm{H}_y=\frac{k_1{}^2c}{i\mu_1\omega}\,.\,e^{-ik_1z}\,;\quad \mathrm{E}_y=\mathrm{E}_z=\mathrm{H}_x=\mathrm{H}_z=0. \qquad . \;.\;. \quad (106)$$

This plane wave is polarized in the y direction, but we shall later on have to take the mean square values over all directions of polarizations at right angles to the z-axis in order to account for the unpolarized light from the sun. The two parts of the Hertzian vector, introduced by Debye, necessary to describe this field are

$$\Pi_{\mathrm{tot}}=\Pi_{\mathrm{tot,\,el}}+\Pi_{\mathrm{tot,\,magn}},$$

and the components E and H of the electromagnetic field are derived from this wave potential as

$$\left.\begin{aligned}\mathrm{E}&=\mathrm{curl\ curl}\,(r\Pi_{\mathrm{el}})+\frac{i\mu\omega}{c}\,.\,\mathrm{curl}\,(r\Pi_{\mathrm{magn}}),\\ \mathrm{H}&=\frac{ck^2}{i\mu\omega}.\,\mathrm{curl}\,(r\Pi_{\mathrm{el}})+\mathrm{curl\ curl}\,(r\Pi_{\mathrm{magn}}).\end{aligned}\right\}. \quad (107)$$

The proper primary field (106) is then obtained from (107) by taking $\Pi_{\mathrm{pr}}=\Pi_{\mathrm{pr,\,el}}+\Pi_{\mathrm{pr,\,magn}}$, where

$$\left.\begin{aligned}\Pi_{\mathrm{pr,\,el}}&=\frac{\cos\phi}{ik_1r\sin\vartheta}\\ &\quad .\{e^{-ik_1r\cos\vartheta}-\cos(k_1r)+i\sin(k_1r)\cos\vartheta\},\\ \Pi_{\mathrm{pr,\,magn}}&=\frac{ic}{\mu_1\omega}\cdot\frac{\sin\phi}{r\sin\vartheta}\\ &\quad .\{e^{-ik_1r\cos\vartheta}-\cos(k_1r)+i\sin(k_1r)\cos\vartheta\}.\end{aligned}\right\}. \quad (108)$$

Through this artifice the Hertzian vector has a radial component only given by $r\Pi_{\mathrm{tot}}$.

In order again to fulfil the boundary conditions we want instead of (108) the equivalent harmonic series

$$\left.\begin{aligned}\Pi_{\mathrm{pr,\,el}}&=-\cos\phi\\ &\quad .\sum_{n=1}^{\infty}(2n+1)i^{-n}\psi_n(k_1r)\,.\,\mathrm{P}_n{}^{1}(\cos\vartheta),\\ \Pi_{\mathrm{pr,\,magn}}&=-\sin\phi\\ &\quad .\sum_{n=1}^{\infty}(2n+1)i^{-n}\psi_n(k_1r)\,.\,\mathrm{P}_n^{-1}(\cos\vartheta).\end{aligned}\right\}. \quad (109)$$

As is also well known in the plane problem the boundary conditions are different for Π_{el} and Π_{magn}. So far in this paper we have used the electric boundary conditions only, whereas now we have to fulfil the two separate conditions

$$\frac{\partial}{\partial r}(r\Pi_{\text{tot, el}}) \text{ and } k^2\Pi_{\text{tot, el}} \text{ continuous at } r=a,$$

$$\frac{\partial}{\partial r}(r\Pi_{\text{tot, magn}}) \text{ en } \Pi_{\text{tot, magn}} \text{ continuous at } r=a.$$

Extending the results of § 10, where we derived the spherical reflexion coefficients R_{11} and R_{12}, which we shall further call $R_{11,\text{el}}$ and $R_{12,\text{el}}$, we shall have to introduce as well $R_{11,\text{magn}}$ and $R_{12,\text{magn}}$, which are at once derived from the magnetic boundary conditions, and which are found to be

$$R_{11,\text{magn}} = \frac{-\left[x\frac{d}{dx}\log\{x\zeta_n^{(2)}(x)\}\right]_{x=k_1a} + \left[x\frac{d}{dx}\log\{x\zeta_n^{(2)}(x)\}\right]_{x=k_2a}}{\left[x\frac{d}{dx}\log\{x\zeta_n^{(1)}(x)\}\right]_{x=k_1a} - \left[x\frac{d}{dx}\log\{x\zeta_n^{(2)}(x)\}\right]_{x=k_2a}},$$

$$R_{12,\text{magn}} = 1 + R_{11,\text{magn}}.$$

All the results obtained heretofore for the electric dipole may at once be applied in slightly extended form for the present rainbow case. The essential differences are :

1. Here we have $\Pi_{\text{el}}+\Pi_{\text{magn}}$ instead of Π_{el} only.
2. Both the source Q and the observer P are at infinite distance from the centre of the sphere (raindrop).
3. The occurrence of $\left.\begin{matrix}\cos\phi\\ \sin\phi\end{matrix}\right\}.P_n^{-1}(\cos\vartheta)$ instead of $P_n^0(\cos\vartheta)$.

Now the rainbows of different order K are rigorously given by the subsidiary waves O_K^e. Thus we find instead of (70 *d*) the series

$$(O_K^e)_{\text{el}} = -\tfrac{1}{2}\cos\phi \cdot \sum_{n=1}^{\infty}(2n+1)i^{-n}.(R_{12}R_{22}^K R_{21})_{\text{el}} \cdot \left\{\frac{\zeta_n^{(1)}(k_2a)}{\zeta_n^{(2)}(k_2a)}\right\}^{K+1} \cdot \frac{\zeta_n^{(2)}(k_1a)}{\zeta_n^{(1)}(k_1a)} \cdot \zeta_n^{(1)}(k_1r).P_n^{-1}(\cos\vartheta),$$

and an analogous series for the magnetic contribution.

As, moreover, the actual bows are concentrated near

the minimum of deviation (*i. e.*, the caustic), we have, as explained in § 16, for numerical evaluation to use at least the third-order approximation for the elementary waves $O^e_{K,0}$, for which the electric part is easily found to be

$$(O^e_{K,0})_{el} \sim \frac{e^{iS_K}}{ik_1 r} \cdot \sqrt{\frac{\pi}{6k_1 a \cdot \sin\tau_2 \cdot \sin\vartheta}} \cdot (R_{12} R^K_{22} R_{21})_{el} \cdot \cos\phi \cdot \frac{dD}{dn} \Big/ \frac{d^2D}{dn^2} \cdot e^{\frac{i}{3}\left(\frac{dD}{dn}\right)^3 \Big/ \left(\frac{d^2D}{dn^2}\right)^2} \cdot H^{(1)}_{1/3}\left\{ \frac{\left(\frac{dD}{dn}\right)^3}{3\left(\frac{d^2D}{dn^2}\right)^2} \cdot e^{-i\pi} \right\}, \quad (110)$$

which is the equivalent of (95) and where the derivatives of $D(n)$ are at once found from its definition (103). The expression for the corresponding magnetic part is again quite analogous.

In order, further, to obtain the actual field components from the wave potential (110) we apply (107), bearing in mind that only S_K has to be differentiated, because $ka >> 1$. Now the physiological intensity of the observed light is usually considered as given by the E vector * and can therefore be represented by $I = \overline{E^2}$, where the dash over the E^2 means the value averaged over all polarization directions ϕ. Now $E_r = 0$, and we are left with E_ϑ, corresponding to the component at right angles ($\perp$) to the arc of the rainbow, and E_ϕ, which corresponds to the component along the arc ($\parallel$) of the bow. Calling further I_0 the intensity of the E_0 vector of the incident light, we are interested in

$$\frac{I_\parallel}{I_0} = \frac{\overline{E_\phi^2}}{\overline{E_0^2}} \left(= \frac{\overline{H_\vartheta^2}}{\overline{H_0^2}} \right),$$

$$\frac{I_\perp}{I_0} = \frac{\overline{E_\vartheta^2}}{\overline{E_0^2}} \left(= \frac{\overline{H_\phi^2}}{\overline{H_0^2}} \right).$$

We thus obtain from (110):

(a) *the absolute intensity*

$$\frac{I_\parallel}{I_0} \sim \frac{\pi}{12} \cdot \frac{(R_{12} R^K_{22} R_{21})_{el}^2}{k_1 a \cdot (k_1 r)^2 \cdot \sin\tau_2 \sin^3 D} \cdot \left(\frac{dD}{dn} \Big/ \frac{d^2D}{dn^2} \right)^2 \cdot \left[H^{(1)}_{1/3}\left\{ \frac{\left(\frac{dD}{dn}\right)^3}{3\left(\frac{d^2D}{dn^2}\right)^2} \cdot e^{-i\pi} \right\} \right]^2, \quad (111)$$

* M. Born, 'Optik' (Berlin, 1933), p. 24.